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ON A FIXED POINT THEOREM ON UNIFORMLY CONVEX BANACH SPACES

P. VEERAMANI AND D. V. PAI
Department of Mathematics, Indian Institute of Technology, Powai, Bombay 400076

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In this paper a fixed point theorem is proved by using normal structure. This generalizes results of Kannan, Ćirić and Rhoades.

1. INTRODUCTION

A result of continuing interest in fixed point theory is the one due to Kirk (1965). This states that a nonexpensive self-mapping of a bounded closed and convex subset possessing normal structure in a reflexive Banach space has a fixed point. Recently Kannan (1971) and Ćirić (1975) have obtained results in basically the same spirit by suitably modifying the non-expensive condition on the mapping and the condition of normal structure on the underlying set. In this paper by using normal structure we obtain a fixed point theorem which generalizes results of Kannan (1971), Ćirić (1975) and Rhoades (1977).

2. FIXED POINT THEOREMS IN BANACH SPACES

Let S be a bounded subset of a Banach space X. A point $x_0 \in S$ is said to be a 'non-diametral' point of S if $\sup\{\|x-x_0\| : x \in S\} < \delta(S)$ (where $\delta(S) = \sup\{\|x-y\| : x,y \in S\}$, the diameter of S).

A bounded closed convex subset K of a Banach space X is said to have 'normal structure' if for each closed convex subset H of K which contains more than one point there exists an $x \in H$ which is a non-diametral point of H.

Evidently, a bounded closed convex subset K of a Banach space X has 'normal structure' if and only if for each closed convex subset H of K which contains more than one point there exists an $x \in H$ and $\alpha(H)$, $0 < \alpha(H) < 1$, such that $\sup\{\|x-y\| : y \in H\} = r_x(H) \leqslant \alpha(H)\delta(H)$.

Theorem 2.1—Let K be a nonempty weakly compact convex subset of the Banach space X. Assume K has normal structure. Let T_1, T_2 be mappings of K into itself satisfying:

(i) for each closed convex subset F of K invariant under T_1 and T_2 there exists some $\alpha_1(F), 0 \leqslant \alpha_1(F) < 1$, such that

$$\|T_1x - T_2y\| \leqslant \max\{\tfrac{1}{2}(\|x-T_1x\| + \|y-T_2y\|), \tfrac{1}{2}(\|x-T_2y\| + \|y-T_1x\|), \tfrac{1}{3}(\|x-y\| + \|x-T_1x\| + \|y-T_2y\|), r_x(F), \alpha_1\delta(F)\}$$

for $x,y \in F$;

(ii) $T_1 C \subset C$ if and only if $T_2 C \subset C$ for each closed convex subset C of K;

(iii) for each closed convex subset D of K invariant under T_1 and T_2 there exists some $\alpha_2(D)$, $\frac{1}{2} \leqslant \alpha_2(D) < 1$, such that either

$$\sup_{z \in D} \| z - T_1 z \| \leqslant \max\{r(D), \ \alpha_2 \delta(D)\}$$

$$(r(D) = \inf \ \{r_x(D) \ ; \ x \in D)$$

or

$$\sup_{z \in D} \| x - T_2 z \| \leqslant \max \{r(D), \ \alpha_2 \delta(D)\}$$

$$(r(D) = \inf \ \{r_x(D) : x \in D\}).$$

Then there exists a common fixed point of T_1 and T_2.

PROOF: We imitate in parts the proof of Kirk's theorem. Let $\mathscr{F}$ denote the family of all nonempty closed convex subsets of K, each of which is mapped into itself by T_1 and T_2. Ordering $\mathscr{F}$ by set inclusion, by weak compactness of K and Zorn's lemma, we obtain a minimal element F of K. By the definition of normal structure, there exists $x_0 \in F$ such that

$$\sup\{ \| x_0 - y \| : y \in F\} = r_{x_0}(F) \leqslant \alpha_3 \ \delta(F)$$

for some α_3, $0 < \alpha_3 < 1$.

Without loss of generality assume that

$$\sup_{z \in F} \| z - T_2 z \| \leqslant \max\{r(F), \alpha_2 \ \delta(F)\}$$

for some α_2, $\frac{1}{2} \leqslant \alpha_2 < 1$. If

$$\begin{aligned} \| T_1 x - T_2 y \| \leqslant \max\{ & \tfrac{1}{2}(\| x - T_1 x \| + \| y - T_2 y \|), \\ & \tfrac{1}{3}(\| x - T_2 y \| + \| y - T_1 x \|), \\ & \tfrac{1}{3}(\| x - y \| + \| x - T_1 x \| + \| y - T_2 y \|), \\ & r_x(F)\} \end{aligned}$$

for all $x,y \in F$,

let $\beta = \max\{\alpha_2, \alpha_3\}$

and $F_\delta = \{x \in F : r_x(F) \leqslant \beta \ \delta \ (F)\}$.

Otherwise, by hypothesis (i) there exists $\alpha_1(F)$, $0 \leqslant \alpha_1(F) < 1$, such that

$$\| T_1 x - T_2 y \| \leqslant \alpha_1 \ \delta(F) \text{ for some } x,y \in F.$$

Let $\beta = \max\{\alpha_1, \alpha_2, \alpha_3\}$

and $F_\delta = \{x \in F : r_x(F) \leqslant \beta \ \delta(F)\}$.

As $x_0 \in F_\delta$, F_δ is nonempty.

Evidently, F_δ is convex. Since $x \to r_x(F)$ is continuous, F_δ is closed.

Let $x \in F_\delta$. Then

$$\begin{aligned} \| T_1 x - T_2 y \| \leqslant \max \ \{ & \tfrac{1}{2}(\| x - T_1 x \| + \| y - T_2 y \|), \\ & \tfrac{1}{3}(\| x - T_2 y \| + \| y - T_1 r \|), \\ & \tfrac{1}{3}(\| x - y \| + \| x - T_1 x \| + \| y - T_2 y \|), \\ & r_x(F), \ \alpha_1 \ \delta(F)\} \\ \leqslant \beta \delta(F) & \text{ for } y \in F. \end{aligned}$$

This gives that $T_1(F)$ is contained in a spherical ball $\overline{U}$ centred at T_1x and of radius $\beta\,\delta\,(F)$, i. e., $T_2(F) \subset \overline{U}$, whence $T_2(F \cap \overline{U}) \subset F \cap \overline{U}$ and by hypothesis (ii) $T_1(F\cap\overline{U})\subset F\cap\overline{U}$. By the minimality of F, we obtain $F\subset\overline{U}$. Hence $r_{T_1x}(F)\leqslant\beta\delta F$, and this implies $T_1x\in F_\delta$. Therefore, $T_1(F_\delta)\subset F_\delta$ and by hypothesis (ii) $T_2(F_\delta)\subset F_\delta$. Hence, $F_\delta\in\mathscr{F}$. But $\delta(F_\delta)\leqslant\beta\delta(F)<\delta(F)$, which contradicts the minimality of F. Hence, F contains a unique x_0 such that $T_1x_0=x_0=T_2x_0$.

Remark: If we replace (i) by

$$\text{(i}')\ \| T_1x-T_2y \| \leqslant \max\{\tfrac{1}{2}(\| x-T_1x \| + \| y-T_2y \|),\ \tfrac{1}{3}(\| x-T_2y \| + \| y-T_1x \|),\ \tfrac{1}{3}(\| x-y \| + \| x-T_1x \| + \| y-T_2y \|)\}$$

$(x,y\in K)$ in the above theorem, then there exists a unique point x_0 in K such that $T_1x_0=x_0=T_2x_0$.

The following lemma is known (Belluce and Kirk 1967, Theorem 4.1).

Lemma 2.2—If K is a bounded closed convex subset of a uniformly convex Banach space, then K has normal structure.

The following theorem generalizes results of Kannan (1971, Theorem 2), Ćirić (1975, Theorem 2) and Rhoades (1977, Theorem 1).

Theorem 2.3—Let K be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Let T_1,T_2 be mappings of K into itself satisfying

$$\text{(i)}\ \| T_1x-T_2y \| \leqslant \max\{\tfrac{1}{2}(\| x-T_1x \| + \| y-T_2y \|),\ \tfrac{1}{3}(\| x-T_2y \| + \| y-T_1x \|),\ \tfrac{1}{3}(\| x-y \| + \| x-T_1x \| + \| y-T_2y \|)\}$$

for $x, y \in K$;

and (ii) and (iii) of Theorem 2.1.

Then the sequence $\{x_n\}$ of iterates defined by

(iv) $x_0 \in K$,

(v) $y_n=(1-\beta_n)x_n+\beta_nT_1x_n,\ n \geqslant 0$,

(vi) $x_{n+1}=(1-\alpha_n)x_n+\alpha_nT_2y_n,\ n \geqslant 0$;

with $\{\alpha_n\}$, $\{\beta_n\}$ satisfying

(i′) $0\leqslant\alpha_n,\ \beta_n\leqslant 1$ for all n,

(ii′) $\sum_n \alpha_n(1-\alpha_n)=\infty$, and

(iii′) $\overline{\lim}\ \beta_n=\beta<1$,

converges to the unique common fixed point of T_1 and T_2.

Proof: Existence of fixed point follows from Theorem 2.1. The remaining proof is similar to that of Pai and Veeramani (1981, Theorem 4.1) and hence is omitted.

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