*Indian J. pure appl. Math.*, **13**(6): 651–655, June 1982

# ON CHAINS OF GENERALIZED KUMMER'S TRANSFORM OF TWO VARIABLES

P. N. INDURKAR AND R. K. SAXENA

*Department of Mathematics, Shivaji University, Kolhapur* 416004

(*Received* 10 *July* 1981)

In this paper, we have established a theorem on chains of generalized Kummer's transform of two variables. The results on specialising the parameters suitably, yield as particular cases, interesting results for Kummer's transform and Laplace transform of two variables.

## 1. INTRODUCTION

The well known Laplace transform of two variables (Ditkin and Prudnikor 1962) has been introduced by the integral equation

$$f(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy} F(x, y)\, dx\, dy,\ \text{Re}(p, q) > 0 \qquad \ldots(1.1)$$

and denoted by $f(p, q) \doteqdot F(x, y)$.

A generalization of (1.1) has been recently introduced by the authors (Indurkar and Saxena 1981) in the form

$$f(p, q) = \int_0^\infty \int_0^\infty G^{1,m}_{m,m+1}\left(x p \middle/ \begin{matrix} 1-\alpha_m \\ 0,\ 1-\beta_m \end{matrix}\right) G^{1,n}_{n,n+1}\left(q y \middle/ \begin{matrix} 1-\gamma_n \\ 0,\ 1-\delta_n \end{matrix}\right) F(x, y)\, dx\, dy \qquad \ldots(1.2)$$

we shall call (1.2) as generalized Kummer's transform of two variables and symbolically denote it by

$$f(p, q) = G_k\left[F(x, y) : \begin{matrix} 1-\alpha_m;\ 0,\ 1\ \ \beta_m \\ 1-\gamma_n;\ 0,\ 1-\delta_n \end{matrix}\right] = G_k[F(x, y)]$$

where $G^{m,n}_{p,q}\left(x \middle/ \begin{matrix} a_r \\ b_s \end{matrix}\right)$ denotes Meijer's $G$ function (Meijer 1941)

When $m = n = 1$ (1.2) reduces to

$$f(p, q) = \frac{\Gamma(\alpha_1)\,\Gamma(\gamma_1)}{\Gamma(\beta_1)\,\Gamma(\delta_1)} \int_0^\infty \int_0^\infty {}_1F_1(\alpha_1;\ \beta_1;\ -px)\ {}_1F_1(\gamma_1;\ \delta_1;\ -qy)\ F(x, y)\, dx\, dy \qquad \ldots(1.3)$$

called Kummer's transform of two variables introduced by Vyas and Saxena (1969) and which is symbolically denoted by

$$f(p, q) = K[F(x, y) : \alpha_1, \beta_1; \gamma_1, \delta_1] = K[F(x, y)].$$

When $m = n = 1$, $\alpha_1 = \beta_1$, $\gamma_1 = \delta_1$ (1.2) reduces to (1.1).

In this paper, we have obtained a chain of generalized Kummer's transform of two variables defined in (1.2). In what follows the symbol $\Delta\,(n,\,a)$ denotes the set of parameters $\frac{a}{n}, \frac{a+1}{n}, \frac{a+2}{n}, \ldots, \frac{a+n-1}{n}$ where $n$ is a positive integer and the symbol $\Delta\,((n,\,n_p))$ denotes the set of parameters $\Delta\,(n,a_1),\ \Delta\,(n,\,a_2),\ \ldots,\ \Delta\,(n,\,a_p)$.

## 2. Theorem

If

$$f_1\,(p,q) = G_k\,[\,F(x,y)\,], \qquad \ldots(2.1)$$

$$f_2\,(p,q) = G_k\,[\,(xy)^{-1/2} f_1\,(\tfrac{1}{x}, \tfrac{1}{y})\,], \qquad \ldots(2.2)$$

$$f_3\,(p,q) = G_k\left[\frac{4}{\pi}\,(xy)^{1/2} f_2\left(\frac{1}{4x^2}, \frac{1}{4y^2}\right)\right], \qquad \ldots(2.3)$$

$$f_4\,(p,q) = G_k\left[\frac{4}{\pi}\,(x\,y)^{1/2} f_3\left(\frac{1}{4x^2}, \frac{1}{4y^2}\right)\right], \qquad \ldots(2.4)$$

---

$$f_r\,(p,q) = G_k\left[\frac{4}{\pi}\,(x\,y)^{1/2} f_{r-1}\left(\frac{1}{4x^2}, \frac{1}{4y^2}\right)\right], \qquad \ldots(2.5)$$

then

$$f_r\left(\frac{p^2}{4}, \frac{q^2}{4}\right) = (2)^{5r-10\alpha+2r(4\alpha-r)-4r}\ \prod_{r=2}^{r}[\alpha]^{\sum_{j=1}^{m}(\alpha_j-\beta_j)+\sum_{j=1}^{n}(\gamma_j-\delta_j)}$$

$$\times\ (\pi)^{2-2\alpha}\ (p\,q)^{3-4\alpha} \int_0^\infty\!\!\int_0^\infty$$

$$G^{2\alpha m,2\alpha}_{2\alpha(m+1),2\alpha m}\left(\left(\frac{2\,\alpha}{p\,x}\right)^{2\alpha}\middle/\begin{matrix}1,\ \Delta\,(1,\tfrac{1}{2}),\ \ldots\ \Delta\left(\alpha, \dfrac{5-4\,\alpha}{2}\right),\ \Delta\,((1,\beta_m-\tfrac{1}{2})),\\ \ldots,\ \Delta\,((\alpha,\,\beta_m+3/2-2\,\alpha));\ \Delta\,((1,\,\beta_m))\\ \Delta\,((1,\,\alpha_m)),\ \Delta\,((1,\,\alpha_m-\tfrac{1}{2})),\ \ldots,\ \Delta\,((\alpha,\,\alpha_m+\tfrac{3}{2}-2\alpha))\end{matrix}\right)$$

$$G^{2\alpha n,\,2\alpha}_{2\alpha(n+1),2\alpha n}\left(\left(\frac{2\,\alpha}{q\,y}\right)^{2\alpha}\middle/\begin{matrix}1,\ \Delta\,(1,\tfrac{1}{2}),\ \ldots,\ \Delta\left(\alpha, \dfrac{5-4\,\alpha}{2}\right),\ \Delta\,((1,\,\delta n-\tfrac{1}{2})),\ldots,\\ \Delta\,((\alpha,\,\delta n+3/2-2\alpha));\ \Delta\,((1,\,\delta n))\\ \Delta\,((1,\,\gamma_n)),\ \Delta\,((1,\,\gamma_n-\tfrac{1}{2})),\ldots,\ \Delta\,((\alpha,\,\gamma_n+\tfrac{3}{2}-2\alpha))\end{matrix}\right)$$

$$(x\,y)^{2\alpha-1}\ F(x^{2\alpha}, y^{2\alpha})\ dx\,dy. \qquad \ldots(2.6)$$

provided generalised Kummer's transform of $\left|\,F(x,y)\,\right|$, $\left|\,(x,y)^{\frac{1}{2}} f_1\left(\frac{1}{x}, \frac{1}{y}\right)\right|$ and $\left|\frac{4}{\pi}\,(xy)^{\frac{1}{2}}\ f_n\left(\frac{1}{4x^2}, \frac{1}{4y^2}\right)\right|$ where $n=2, 3, \ldots, r-1$, all exist, the integral in (2.6) is absolutely convergent, $\left|\,\arg p\,\right| < \frac{1}{2}\pi$, 1 $\left|\,\arg q\,\right| < \frac{1}{2}\pi$, Re $(p,\,q) > 0$, here $\alpha$ stands for $2^{r-2}$ and $\prod_{r=2}^{r}$ indicates the product of factors within the bracket for $r=2$ to any integral value of $r$.

PROOF : We have from (2.1),

$$f_1(p,q)=\int_0^\infty\int_0^\infty G^{1,m}_{m,m+1}\left(px\,\Big/\,{1-\alpha_m \atop 0,1-\beta_m}\right)G^{1,n}_{n,n+1}\left(qy\,\Big/\,{1-\gamma_n \atop 0,1-\delta_n}\right)F(x,y)\,dx\,dy.$$

Substituting the expression for $f_1\left(\frac{1}{x},\frac{1}{y}\right)$ from (2.1) in (2.2), using a known result [ Erdélyi *et al.* 1953, p. 209 (9) ], interchanging the order of integration which is justified due to absolute convergence of the integrals involved and evaluating the inner double integral with help of a known result (Saxena 1960, p. 401), we get

$$f_2\left(\frac{p^2}{4},\frac{q^2}{4}\right)=\frac{2^4}{pq}\int_0^\infty\int_0^\infty$$

$$\times\ G^{2m,2}_{2m+2,2m}\left(\left(\frac{2}{px}\right)^2\Big/\,{1,\Delta(1,\frac{1}{2}),\Delta((1,\beta_m-\frac{1}{2})),\Delta((1,\beta_m)) \atop \Delta((1,\alpha_m)),\Delta((1,\alpha_m-\frac{1}{2}))}\right)$$

$$\times\ G^{2n,2}_{2n+2,2n}\left(\left(\frac{2}{qy}\right)^2\Big/\,{1,\Delta(1,\frac{1}{2}),\Delta((1,\delta n-\frac{1}{2})),\Delta((1,\delta n)) \atop \Delta((1,\gamma_n)),\Delta((1,\gamma_n-\frac{1}{2}))}\right)$$

$$xy\,F(x^2,y^2)\,dx\,dy.$$

Now using the above expression in (2.3) and proceeding as above, we obtain

$$f_3\left(\frac{p^2}{4},\frac{q^2}{4}\right)=2^{21}\,\pi^{-2}\,(p\,q)^{-5}\,(2)^{\sum_{j=1}^{m}(\alpha_j-\beta_j)+\sum_{j=1}^{n}(\gamma_j-\delta_j)}$$

$$\times\int_0^\infty\int_0^\infty G^{4m,4}_{4m+4,4m}\left(\left(\frac{4}{px}\right)^4\Big/\,{1,\Delta(1,\frac{1}{2}),\Delta(2,-\frac{3}{2}),\Delta((1,\beta_m-\frac{1}{2})),\Delta((2,\beta_m-\frac{5}{2}),\ \Delta((1,\beta_m)) \atop \Delta((1,\alpha_m)),\Delta((1,\alpha_m-\frac{1}{2})),\Delta((2,\alpha_m-\frac{5}{2}))}\right)$$

$$\times\ G^{4n,4}_{4n+4,4n}\left(\left(\frac{4}{qy}\right)^4\Big/\,{1,\Delta(1,\frac{1}{2}),\Delta(2,-\frac{3}{2}),\Delta((1,\delta_n-\frac{1}{2})),\Delta((2,\delta_n-\frac{5}{2})),\Delta((1,\delta_n)) \atop \Delta((1,\gamma_n)),\Delta((1,\gamma_n-\frac{1}{2})),\Delta((2,\gamma_n-\frac{5}{2}))}\right)$$

$$x^3\,y^3\,F(x^4,y^4)\,dx\,dy$$

repeating the process successively with correspondences (2.4), ... , respectively, we arrive at the result (2.6)

## 3. PARTICULAR CASES

(a) Substituting $m=n=1$, $\alpha_1=\beta_1$, $\gamma_1=\delta_1$ in the above theorem, we get

If $f_1(p,q)=F(x,y)$ ...(3.1)

$$f_2(p,q)=(x\,y)^{1/2}f_1\left(\frac{1}{x},\frac{1}{y}\right) \qquad ...(3.2)$$

$$f_3(p,q)=\frac{4}{\pi}(x\,y)^{-1/2}f_2\left(\frac{1}{4x^2},\frac{1}{4y^2}\right) \qquad ...(3.3)$$

$$f_4(p,q)=\frac{4}{\pi}(x\,y)^{1/2}\,f_3\left(\frac{1}{4x^2},\frac{1}{4y^2}\right), \qquad ...(3.4)$$

— — — — — — — — — — — — — — —

$$f_r(p,q)=\frac{4}{\pi}(xy)^{1/2}f_{r-1}\left(\frac{1}{4x^2},\frac{1}{4y^2}\right), \qquad ...(3.5)$$

then

$$f_r\left(\frac{p^2}{4}, \frac{q^2}{4}\right) = 2^{5r-10\alpha+2r(4\alpha-r)-4r} (\pi)^{2-2\alpha} (pq)^{3-4\alpha}$$

$$\times \int_0^\infty \int_0^\infty G^{0,2\alpha}_{2\alpha,0}\left(\left(\frac{2\alpha}{px}\right)^{2\alpha} \Big/ 1, \Delta(1, \tfrac{1}{2}), \Delta(2, -\tfrac{3}{2}), \ldots, \Delta\left(\alpha, \frac{5-4\alpha}{2}\right)\right)$$

$$\times G^{0,2\alpha}_{2\alpha,0}\left(\left(\frac{2\alpha}{qy}\right)^{2\alpha} \Big/ 1, \Delta(1, \tfrac{1}{2}), \Delta(2, -\tfrac{3}{2}), \ldots, \Delta\left(\alpha, \frac{5-4\alpha}{2}\right)\right)$$

$$\times (xy)^{2\alpha-1} F(x^{2\alpha}, y^{2\alpha})\, dx\, dy. \qquad \ldots(3.6)$$

provided Laplace transforms of $\left| F(x,y) \right|$, $\left| (x, y)^{-1/2} f_1\left(\frac{1}{x}, \frac{1}{y}\right) \right|$ and $\left| \frac{1}{\pi}(xy)^{1/2} f_n\left(\frac{1}{4x^2}, \frac{1}{4y^2}\right) \right|$ where $n = 2, 3, \ldots (r-1)$ all exist, the integral $f_r\left(\frac{p^2}{4}, \frac{q^2}{4}\right)$ is absolutely convergent, $|\arg p| < \frac{1}{2}\pi$, $|\arg q| < \frac{1}{2}\pi$, $\text{Re}(p, q) > 0$, here $\alpha$ stands for $2r^{2-}$.

(b) Substituting $m = n = 1$ in the main theorem we obtain a corresponding result in Kummer's transform of two variables.

## 4. Application

In particular case (a) of the above Theorem, taking

$$F(x, y) = G^{h,l}_{u,v}\left(zxy \Big/ \begin{matrix}(c_u)\\(d_v)\end{matrix}\right)$$

where $u + v < 2(h + l)$, $|\arg z| < \left((h + 1 - \frac{u}{2} - \frac{v}{2}\right)\pi$,

Then using [ Erdélyi *et al.* 1954, p. 222 (34) ], we have

$$f_1(p, q) = (pq)^{-1} G^{h,l+2}_{u+2,v}\left(\frac{z}{pq} \Big/ \begin{matrix}0, 0, (c_u)\\(d_v)\end{matrix}\right)$$

$$\text{Re}\, d_j + 1 < 0,\ 1 \leqslant j \leqslant h,\ \text{Re}(p, q) > 0.$$

and

$$f_2(p, q) = (pq)^{-3/2} G^{h,l+4}_{u+4,v}\left(\frac{z}{pq} \Big/ \begin{matrix}\frac{-1}{2}, \frac{-1}{2}, 0, 0, (c_u)\\(dv)\end{matrix}\right)$$

and now utilizing a known result [Saxena 1960, p. 401 ], we obtain

$$f_3(p, q) = \frac{2^{15}}{\pi^2}(pq)^{-9/2} G^{h,l+8}_{u+8,v}\left(\frac{2^8 z}{p^2q^2} \Big/ \begin{matrix}(2, \frac{-7}{2}, (2, \frac{-7}{2}), \frac{-1}{2}, \frac{-1}{2}, 0, 0\,(c_u)\\(dv)\end{matrix}\right)$$

and thus proceeding successively, we get

$$f_r(p, q) = (2)^{3r+4-26\alpha+12\alpha r-2r^2} (\pi)^{2-2\alpha} (pq)^{-3/2(2\alpha-1)}$$

$$\times G^{h,l+4\alpha}_{u+4\alpha,v}\left(\frac{\alpha^{4\alpha} z}{(pq)^\alpha} \Big/ \begin{matrix}\left(2\alpha, \frac{5-6\alpha}{2}\right)\left(2\alpha, \frac{5-6\alpha}{2}\right), \ldots, \left(2, \frac{-7}{2}\right), \left(2, \frac{-7}{2}\right), \frac{-1}{2}, \frac{-1}{2} 0, 0\,(C_u)\\(dv)\end{matrix}\right)$$

where $\alpha = 2^{r-2}$.

Hence

$$f_r\left(\frac{p^2}{4}, \frac{q^2}{4}\right) = (2)^{3r-2-14\alpha+12\alpha r-2r^2} (\pi)^{2-2\alpha} (pq)^{-3(2\alpha-1)}$$

$$G^{h,\, l+4\alpha}_{u+4\alpha,\, v}\left(\frac{(2\alpha)^{4\alpha} z}{(pq)^{-\alpha}} \middle/ \begin{matrix}\left(2\alpha, \frac{5-6\alpha}{2}\right), \left(2\alpha, \frac{5-6\alpha}{2}\right), \ldots, \left(2, \frac{-7}{2}\right), \left(2, \frac{-7}{2}\right), \frac{-1}{2}, \frac{-1}{2}\, 0, 0, (C_u) \\ (d_v)\end{matrix}\right)$$

provided $\mathrm{Re}\, d_j + 1 < 0$, $1 \leqslant j \leqslant h$, $\mathrm{Re}\,(p, q) > 0$.

## REFERENCES


Ditkin, V. A., and Prudnikov, A. P. (1962). Operational Calculus in two Variables. Pergamon Press, London.

Erdelyi, A. *et al.*, (1953) Higher Transcedental Functions, Vol, I. Mc-Graw Hill Book Co., Inc., N.Y.

——— (1954). Tables of Integral Transforms Vol I, Mc-Graw Hill Book Co., Inc., N. Y.

Indurkar, P. N., and Saxena, R. K. (1981). Generalized Kummer's transform of two variables. To appear in the *J. Shivaji Univ. (Science)* **12**.

Meijer, C. S. (1941). Neue Integralderstellungen fur Whittakersche Funktionen (Erste Mitteilung) *Proc. Nederl. Akad. Wetensch*, **44**, 81–92.

Saxena, R. K. (1960). Some theorems on generalized Laplace transform I. *Proc. Natn. Inst. Sci. India*, 26 A, No. 4, 400–13.

Vyas, R. C., and Saxena, R. K. (1969). Kummer's transform of two variables. *Riv. Mat. Univ. Parma* (2), **10** 23–32.