Indian J. pure appl. Math., **13**(6): 656–665, June 1982

IMPACT OF EXTRA FLOW IN A TRANSPORTATION PROBLEM

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(*Received* 8 *July* 1981)

The present paper considers a transportation problem, where, because of extra demand in the markets, the total flow needs to be enhanced, compelling some of the factories to increase their productions in order to be able to meet this extra demand. A related standard transportation problem (RTP) is formulated, and it is shown that to each special basic feasible solution [called corner feasible solution (cfs)] to (RTP), there is a corresponding feasible solution to this 'enhanced-flow problem' [called (EP)]. The optimal solutions to (EP) may be obtained from the optimal solutions to (RTP). The factories where production needs to be increased, as well as the markets where this extra production would be consumed, are also identified. Also considered is an extension, where in addition to the flow being enhanced, the new requirement of each market is also specified.

INTRODUCTION

The standard transportation problem, namely:

Problem (*STP*)

$$\text{Minimize} \sum_I \sum_J c_{ij} x_{ij}$$

$$\text{subject to} \sum_J x_{ij} = a_i, \; i \in I$$

$$\sum_I x_{ij} = b_{ij}, \; j \in J$$

$$x_{ij} \geqslant 0, \quad i \in I, j \in J$$

is concerned with transporting at a minimum cost, a homogeneous commodity from each of the factories indexed by $I = \{1, 2, \ldots, m\}$ to a number of markets indexed by $J = \{1, 2, \ldots, n\}$. The total flow in this problem is $\sum_I a_i = \sum_J b_j$.

Quite frequently, it may so happen that there is an extra demand in the markets for the commodities. In order to meet this extra demand, the factories have to increase their productions. The total flow from the factories to the markets is now enhanced by the amount of extra demand. If $P \; (> \sum_I a_i = \sum_J b_j)$ be the enhanced flow, then because of this flow constraint, the given problem (STP) no longer has the transportation structure. Mathematically stated, the enhanced-flow problem is:

Problem (EP)

$$\text{Minimize} \sum_I \sum_J c_{ij} x_{ij}$$

$$\text{subject to} \sum_J x_{ij} \geqslant a_i,\ i \in I$$

$$\sum_I x_{ij} \geqslant b_j,\ j \in J$$

$$\sum_I \sum_J x_{ij} = P \left(P > \sum_I a_i = \sum_J b_j\right)$$

$$x_{ij} \geqslant 0,\ i \in I,\ j \in J.$$

It may be noted that without the flow constraint, the problem (EP) becomes just one of the problems studied by Appa (1973); where he considers variants of the standard transportation problem in which the factory and/or market constraints are inequalities. But here, the problem is very much different because of the flow constraint. A related standard transportation problem (RTP) is formulated by adding one row corresponding to a fictitious factory and one column corresponding to a fictitious market. The existence of a feasible solution to (EP) corresponding to each corner feasible solution to (RTP) is established. An optimal solution to (EP) is shown to be determined from an optimal solution to (RTP).

A numerical example is given in support of the theory developed.

Remark : In case of an unbalanced problem, that is, when $\sum_I a_i \neq \sum_J b_j$, the formulation of the enhanced-flow problem would be different from that of (EP) here.

Theoretical Development

The analytical formulation of the (RTP) associated with (EP) is :

Problem (RTP)

$$\text{Minimize} \sum_{I'} \sum_{J'} c'_{ij} y_{ij}$$

$$\text{subject to} \sum_{J'} y_{ij} = a'_i,\ i \in I' = I \cup \{m+1\}$$

$$\sum_{I'} y_{ij} = b'_j,\ j \in J' = J \cup \{n+1\}$$

$$y_{ij} \geqslant 0,\ i \in I',\ j \in J'$$

where $a'_i = a_i \ \forall\ i \in I$; $b'_j = b_j \ \forall\ j \in J$; $a'_{m+1} = b'_{n+1} = (P - \sum_I a_i) = (P - \sum_J b_j)$

and $c'_{ij} = c_{ij} \ \forall\ i \in I,\ j \in J$

$$c'_{i,n+1} = \min_{j \in J} c_{ij} \ \forall\ i \in I$$

$$c'_{m+1,j} = \min_{i \in I} c_{ij} \ \forall\ j \in J$$

$c'_{m+1,n+1} = M$ (M being an arbitrarily large positive quantity).

The diagram depicting (RTP) is

$$\begin{array}{c|cc|c|c|c|} & & & \text{Fictitious Market} & & \\ \text{Fictitious factory} & c'_{ij} = c_{ij} & & c'_{i,n+1} = \min_{j \in J} c_{ij} & = & a'_i = a_i \\ & c'_{m+1,j} \ \min_{i \in I} c_{ij} & & c'_{m+1,n+1} = M & & a'_{m+1} = P - \sum_I a_i \\ & & = & & & \\ & b'_j = b_j & & b'_{n+1} = P - \sum_I a_i & & \end{array}$$

Definition : Corner-feasible solution (cfs) to (RTP)—A feasible solution $\{y_{ij}\}$, $i \in I', j \in J'$ to (RTP) is called a corner feasible solution if $y_{m+1,n+1} = 0$.

Theorem 1—A corner-feasible solution to (RTP) gives a feasible solution to (EP).

Let $\{y_{ij}\}$, $i \in I'$, $j \in J'$ be a cfs to (RTP)

Define

$$\left.\begin{array}{l} w_{ij} = y_{ij} \ \forall \ i \in I,\ j \in J,\ j \neq p \\ w_{ip} = y_{ip} + y_{i,n+1} \text{ for a single } p \text{ such that} \\ c_{ip} = \min_{j \in J} c_{ij} = c'_{i,n+1} \\ \text{Next, let } x_{ij} = w_{ij} \ \forall \ i \in I,\ j \in J,\ i \neq k \\ x_{kj} = w_{kj} + y_{m+1,j} \text{ for a single } k \text{ such that} \\ c_{kj} = \min_{i \in I} c_{ij} = c'_{m+1,j} \end{array}\right\} \qquad (T)$$

$\{x_{ij}\}$ so defined will be shown to be the feasible solution to (EP) corresponding to the cfs $\{y_{ij}\}$ to (RTP).

Now, for $i \in I$, $(i \neq k)$,

$$\begin{aligned} \sum_I x_{ij} &= \sum_J w_{ij} \\ &= \sum_{J \setminus j \neq p} w_{ij} + w_{ip} \\ &= \sum_{J \setminus j \neq p} y_{ij} + y_{ip} + y_{i,n+1} \\ &= \sum_J y_{ij} + y_{i,n+1} \\ &= \sum_{J'} y_{ij} = a'_i = a_i. \end{aligned}$$

So $\quad \sum_J x_{ij} = a_i \ \forall \ i \in I,\ (I \neq k). \qquad \ldots \qquad (\alpha)$

And for $i = k$, $\sum_I x_{kj} = \sum_J w_{kj} + \sum_J y_{m+1,j}$

$$= \sum_{J|j \neq p} w_{kj} + w_{kp} + \sum_J y_{m+1,j}$$

(*equation continued on p.* 659)

$$= \sum_{J \setminus j \neq p} y_{kj} + y_{kp} + y_{k,n+1} + \sum_{J} y_{m+1,j}$$

$$= \sum_{J} y_{kj} + y_{k,n+1} + \sum_{J} y_{m+1,j}$$

$$= \sum_{J'} y_{kj} + \sum_{J'} y_{m+1,j} \text{ (since } y_{m+1,n+1} = 0 \text{ for cfs)}$$

$$= a_k + a'_{m+1}$$

So $\sum_{J} x_{kj} > a_k \quad \left[\text{since } a'_{m+1} = (P - \sum_i a_i) > 0 \right]$ ...(β)

Combining (α) and (β) we have

$$\sum_{J} x_{ij} \geqslant a_i \ \forall \ i \in I. \quad \text{...(1)}$$

Similarly it can be shown that

$$\sum_{I} x_{ij} \geqslant b_j \ \forall \ j \in J. \quad \text{...(2)}$$

Moreover $\sum_{I} \sum_{J} x_{ij} = \sum_{I \setminus i \neq k} \sum_{J} x_{ij} + \sum_{J} x_{kj}$

$$= \sum_{I / i \neq k} \sum_{J} w_{ij} + \sum_{J} (w_{kj} + y_{m+1,j}) \text{ (using transformation } (T))$$

$$= \sum_{I} \sum_{J} w_{ij} + \sum_{J} y_{m+1,j}$$

$$= \sum_{I} \sum_{J \setminus j \neq p} w_{ij} + \sum_{I} w_{ip} + \sum_{J} y_{m+1,j}$$

$$= \sum_{I} \sum_{J / j \neq p} y_{ij} + \sum_{I} (y_{ip} + y_{i,n+1}) + \sum_{J} y_{m+1,j}$$

$$= \sum_{I} \sum_{J} y_{ij} + \sum_{J} y_{m+1,j} + \sum_{I} y_{i,n+1}$$

$$= \sum_{I} \sum_{J'} y_{ij} + \sum_{J} y_{m+1,j}$$

$$= \sum_{I} \sum_{J'} y_{ij} + \sum_{J'} y_{m+1,j} \quad \text{(since } y_{m+1,n+1} = 0 \text{ for cfs)}$$

$$= \sum_{I'} a'_i + a'_{m+1}$$

$$= \sum_{I} a_i + (P - \sum_{I} a_i) = P.$$

So

$$\sum_{I} \sum_{J} x_{ij} = P. \quad \text{...(3)}$$

Also

$$x_{ij} \geqslant w_{ij} \geqslant y_{ij} \geqslant 0 \ \forall \ i \in I, j \in J.$$

Thus $x_{ij} \geqslant 0 \ \forall \ i \in I, j \in J.$ ...(4)

Relations (1)–(4) show that $\{x_{ij}\}$ is a feasible solution to (EP).

Theorem 2—A non corner feasible solution to (RTP) can never lead to a feasible solution to (EP).

PROOF: Let $\{y_{ij}\}$, $i \in I'$, $j \in J'$ be a non corner feasible solution to (RTP).

So $y_{m+1,n+1} = \lambda \ (> 0)$

So $\sum_J y_{m+1,j} = (P - \sum_I a_i) - \lambda$. ...(5)

Now, for $i \in I$, $\sum_{J'} y_{ij} = a'_i = a_i$

So $\sum_I \sum_{J'} y_{ij} = \sum_I a_i$

That is, $\sum_I \sum_J y_{ij} + \sum_I y_{i,n+1} = \sum_I a_i$. ...(6)

Now, because of the transformation (T), the total flow in the problem (EP)

$$= \sum_I \sum_J y_{ij} + \sum_I y_{i,n+1} + \sum_J y_{m+1,j}$$

$$= \sum_I a_i + \sum_J \bar{y}_{m+1,j} \quad \text{[from (6)]}$$

$$= \sum_I a_i + (P - \sum_I a_i) - \lambda \quad \text{[from (5)]}$$

$$= P - \lambda < P$$

So the total flow in the problem (EP) turns out to be less than P, which is a contradiction.

Hence the non corner feasible solution $\{y_{ij}\}$ to (RTP) can never lead to a feasible solution to (EP).

Note : Provided the problem (RTP) has a cfs, it follows from the definition of $c'_{m+1,n+1}$ that no non corner feasible solution to (RTP) can be its optimal solution.

Theorem 3—The value of the objective function of (RTP) yielded by the corner-feasible solution $\{y_{ij}\}$ is equal to the value of the objective function of (EP) yielded by its corresponding feasible solution $\{x_{ij}\}$.

Proof : The value of the objective function of (RTP) yielded by the corner feasible solution $\{y_{ij}\}$

$$= \sum_{I'} \sum_{J'} c'_{ij}\, y_{ij}$$

$$= \sum_I \sum_{J'} c'_{ij}\, y_{ij} + \sum_{J'} c'_{m+1,j}\, y_{m+1,j}$$

$$= \sum_I \sum_{J'} c'_{ij}\, y_{ij} + \sum_J c'_{m+1,j}\, y_{m+1,j} \quad \text{(since } y_{m+1,n+1} = 0 \text{ for cfs)}$$

$$= \sum_I \sum_J c'_{ij}\, y_{ij} + \sum_I c'_{i,n+1}\, y_{i,n+1} + \sum_J c'_{m+1,j}\, y_{m+1,j}$$

$$= \sum_I \sum_J c_{ij}\, y_{ij} + \sum_I c'_{i,n+1}\, y_{i,n+1} + \sum_J c'_{m+1,j}\, y_{m+1,j}$$

$$= \sum_I \sum_{J \setminus j \neq p} c_{ij}\, y_{ip} + \sum_I c_{ip}\, y_{ip} + \sum_I c_{ip}\, y_{i,n+1} + \sum_J c'_{m+1,j}\, y_{m+1,j}$$

$$= \sum_I \sum_{J \setminus j \neq p} c_{ij}\, w_{ij} + \sum_I c_{ip}\, \omega_{ip} + \sum_J c^{i}_{m+1,j}\, y_{m+1,j} \quad \text{(using transformation } (T))$$

$$= \sum_I \sum_J c_{ij} w_{ij} + \sum_J c^d_{m+1,j} y_{m+1,j}$$

$$= \sum_{I \setminus i j \neq k} \sum_J c_{ij} w_{ij} + \sum_J c_{kj} w_{kj} + \sum_J c_{kj} y_{m+1,j}$$

$$= \sum_{I \setminus j \neq k} \sum_J c_{ij} x_{ij} + \sum_J c_{kj} x_{kj} \quad \text{(using transformation } (T))$$

$$= \sum_I \sum_J c_{ij} x_{ij}$$

= The value of the objective function of (EP) at the corresponding feasible solution $\{x_{ij}\}$.

Conclusion

Optimal corner feasible solutions to (RTP) provide optimal solutions to (EP).

Let $\{\hat{y}_{ij}\}$, $i \in I'$, $j \in J'$ be an optimal cfs to (RTP). Let $\{\hat{x}_{ij}\}$, $i \in I$, $j \in J$ be the corresponding feasible solution to (EP) (Theorem 1). As $\{\hat{y}_{ij}\}$ is an optimal solution to (RTP), the allocations $\hat{y}_{i,n+1}$, $i \in I$ determine optimally the factories i that are to meet the extra demand of the various markets (beyond their minimum requirements). That is, if $\hat{y}_{i,n+1} > 0$ for some $i \in I$, then the pth market, with $c_{ip} = \min_{j \in J} c_{ij} = c'_{i,n+1}$ will be oversupplied by an amount $\hat{y}_{i,n+1}$, by the ith factory. Since $c_{ip} = \min_{j \in J} c_{ij}$, the ith factory could not have oversupplied to any market other than the pth, at a lower cost.

Similarly, the allocation $\hat{y}_{m+1,j}$ in the jth column ($j \in J$) determines, in an optimal way, the factory $k \in I$ with $c_{kj} = \min_{i \in I} c_{ij} = c'_{m+1,j}$ that is to be overburdened by the amount $\hat{y}_{m+1,j}$. As $c_{kj} = \min_{I \in i} c_{ij}$, this extra burden $\hat{y}_{m+1,j}$ could not have been absorbed by any other factory except the kth at a lower cost.

Hence the feasible solution $\{\hat{x}_{ij}\}$ to (EP) corresponding to the optimal cfs $\{\hat{y}_{ij}\}$ to (RTP) is indeed an optimal solution to (EP), yielding the same value of the objective function as the solution $\{\hat{y}_{ij}\}$ (Theorem 3).

Note : If the problem

$$\text{Minimize } \sum_I \sum_J c_{ij} x_{ij}$$

$$\text{subject to } \sum_J x_{ij} \geqslant a_i,\ i \in I$$

$$\sum_I x_{ij} \geqslant b_j,\ j \in J$$

$$x_{ij} \geqslant 0,\ i \in I,\ j \in J$$

is solved by Appa's approach then the optimal solution so obtained may not necessarily lead to an optimal solution to (EP) in case the flow P' in the optimal solution obtained by Appa's approach is such that $P' > P$; and the technique developed in the present paper is mainly suitable to tackle such problems. However, in case $P' < P$, then the optimal solution obtained by Appa's approach will lead to an optimal solution to (EP) by increasing the allocation in the occupied cell with minimum cost, by the amount $(P-P')$.

Numerical Example

For the standard transportation problem

a_i \ b_j	5	2	13	20	
10	6	2	1	3	F_1
6	4	7	8	5	F_2
15	1	2	3	4	F_3
9	7	6	2	2	F_4
	M_1	M_2	M_3	M_4	

let us find a transportation schedule with minimum transportation cost, when the flow is enhanced to $P = 50$.

Note that $(P = 50) > (\Sigma\, a_i = \Sigma\, b_j = 40)$.

The corresponding (EP) is

	$\geqslant 5$	$\geqslant 2$	$\geqslant 13$	$\geqslant 20$
$\geqslant 10$	6	2	1	3
$\geqslant 6$	4	7	8	5
$\geqslant 15$	1	2	3	4
$\geqslant 9$	7	6	2	2

The related transformation problem (RTP) is

	5	2	13	20	10
10	6	2	1	3	1
6	4	7	8	5	4
15	1	2	3	4	1
9	7	6	2	2	2
10	1	2	1	2	M

The optimal solution to (RTP) is

	5	2	13	20	10
10	6	2	1 {10}	3	1
6	4	7	8	5 {6}	4
15	1 {3}	2 {2}	3	4	1 {10}
9	7	6	2	2 {9}	2
10	1 {2}	2	1 {3}	2 {5}	M

So the optimal solution to (EP) is

	$\geqslant 5$	$\geqslant 2$	$\geqslant 13$	$\geqslant 20$	
$\geqslant 10$	6	2	1 {13}	3	F_1
$\geqslant 6$	4	7	8	5 {6}	F_2
$\geqslant 15$	1 {15}	2 {2}	3	4	F_3
$\geqslant 9$	7	6	2	2 {14}	F_4
	M_1	M_2	M_3	M_4	

In the optimal solution to (EP), the production at the factory F_1 is increased by 3, that at F_3 is increased by 2 and at F_4 by 5, whereas at F_2 it remains the same. The market M_1 is the only market where the whole extra production of 10 is being consumed.

EXTENSION

The theory developed in the paper can be further extended to tackle the situation where the flow is to be enhanced to P, and in addition, the new requirement of each market is also specified. So now the flow has to be increased in a particular way so as to meet the new requirements of the markets. The optimal cost in such situations is expected to be atleast equal to the optimal cost obtained when it is just required to enhance the flow to P. So the (EP) in such a situation will be :

$$\text{Minimize} \quad \sum_I \sum_J c_{ij} x_{ij}$$

$$\text{s. t.} \quad \sum_J x_{ij} \geqslant a_i$$

$$\sum_I x_{ij} = \overline{b_j} \; (\geqslant b_j)$$

$$\sum_I \sum_J x_{ij} = P$$

$$x_{ij} \geqslant 0$$

where $\overline{b_j}$ = the new requirement of market j.
The (RTP) associated with this problem is :

$$\text{Minimize} \quad \sum_{I'} \sum_{J} c'_{ij} \; y_{ij}$$

$$\text{s. t.} \quad \sum_{J} y_{ij} = a'_i \; \forall \; i \in I'$$

$$\sum_{I} y_{ij} = \overline{b}_j \; \forall \; j \in J$$

$$y_{ij} \geqslant 0 \; \forall \; i \in I', j \in J$$

where

$$a'_i = a_i \; \forall \; i \in I, \; a'_{m+1} = P - \sum_{I} a_i$$

$$c'_{ij} = c_{ij} \; \forall \; i \in I, \; j \in J; \; c'_{m+1,j} = \min_{i \in I} \; c_{ij} \; \forall \; j \in J.$$

If $\left\{ y^0_{ij} \right\}$, $i \in I'$, $j \in J$ be an optimal solution to (RTP), then the optimal solution $\left\{ x^0_{ij} \right\}$ to this (EP) is given by the transformation :

$$x^0_{ij} = y^0_{ij} \quad \forall \; i \in I, j \in J, i \neq k$$

$$x^0_{kj} = y^0_{kj} + y^0_{m+1,j} \text{ for a single } k \text{ such that}$$

$$c_{kj} = \min_{i \in I} \; c_{ij} = c'_{m+1,j}$$

In the same numerical example given earlier, if the flow is to be enhanced to $P = 50$, and in addition, the new requirements of the markets are given by

$$\overline{b}_1 = 7, \; \overline{b}_2 = 3, \; \overline{b}_3 = 20, \overline{b}_4 = 20$$

then the (EP) in this case is

	7	3	20	20
$\geqslant$ 10	6	2	1	3
$\geqslant$ 6	4	7	8	5
$\geqslant$ 15	1	2	3	4
$\geqslant$ 9	7	6	2	2

The (RTP) associated with the problem is

	7	3	20	20
10	6	2	1	3
6	4	7	8	5
15	1	2	3	4
9	7	6	2	2
10	1	2	1	2

The optimal solution to (RTP) is

	7	3	20	20
10	6	2	1 {10}	3
6	4	7	8	5 {6}
15	1 {7}	2 {3}	3	4 {5}
9	7	6	2	2 {9}
10	1	2	1 {10}	2 {0}

So the optimal solution to the (EP) is

	7	3	20	20
$\geqslant$ 10	6	2	1 {20}	3
$\geqslant$ 6	4	7	8	5 {6}
$\geqslant$ 15	1 {7}	2 (3)	3	4 {5}
$\geqslant$ 9	7	6	2	2 {8}

The optimal cost in this case is 101 which is greater than the optimal cost (90) in the earlier numerical example, where it was just required to enhance the flow to 50.

Acknowledgement

The author is thankful to Prof. R. N. Kaul, University of Delhi, for encouragement during the preparation of the paper.

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