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SLOW VISCOUS ROTATING FLUID PAST A SPHERE

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The governing differential equations for steady, slow viscous rotating fluid past a sphere contain two parameters, namely, $C = 2\omega a/U$ and $Re = Ua/\nu$. Method of matched asymptotic expansions is used for $Re << 1$ and solution is obtained for $CRe \ll 1$. Correction to the drag is calculated and is found to be $D = D_S\,(1+(25\,c^2 Re/9)$ where D_S is Stokes drag. Vorticity components are calculated using the swirl velocity component V_ϕ and streamfunction.

1. INTRODUCTION

It is well known when Whitehead tried to iterate on Stokes solution the condition at infinity could not be satisfied. Oseen (1914) gave a solution, taking into consideration inertial terms, where the condition on the body is satisfied approximately. Proudman and Pearson (1957), Lagerstrom and Kaplun (1957) developed method of matched asymptotic expansions. For the present problem when we tried to iterate on Stokes solution we could not satisfy the condition at infinity just as in well known Whitehead's Paradox. Method of matched asymptotic expansions as developed by Proudman and Pearson is applied to the present investigation. Correction to the Stokes drag and vorticity components are calculated.

2. FORMULATION OF THE PROBLEM

We consider a steady, slow viscous rotating fluid which is in solid body rotation with angular velocity ω about Z axis and moving with uniform velocity U along z-axis. A sphere of radius a is introduced into the flow and kept fixed at the origin. Stokes type of linearisation is applied to governing Navier Stokes equations. After non-dimensionlising the variables as

$$\Psi = \Psi'/a^2\,U,\ \frac{r'}{a} = r,\ 2\omega a/U = C,\ \ Ua/\nu = Re$$

$$v_r = v_r'/U,\ \ v_\theta = v_\phi'/U,\ \ v_\phi = v_\phi'/U$$

where Ψ is the stream function, ν = kinematic viscosity, v_r, v_θ, v_ϕ are velocity components in spherical polar coordinates. The equations are

$$D^4\,\Psi = C\,Re\left(\cos\theta\frac{\partial\Omega}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Omega}{\partial\theta}\right) \qquad \ldots(2\ 1)$$

$$D^2\,\Omega = -C\,Re\left(\cos\theta\ \frac{\partial\Psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Psi}{\partial\theta}\right). \qquad \ldots(2.2)$$

Where $\Omega = r \sin\theta\, v_\phi$ and $D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$

with boundary conditions

$$\Omega = \Psi = \frac{\partial\Psi}{\partial r} = 0 \text{ on } r = 1 \qquad \text{...(2.3)}$$

$$\left.\begin{aligned} \Psi &= \tfrac{1}{2}\, r^2 \sin^2\theta \\ \Omega &= \frac{c}{2}\, r^2 \sin^2\theta \end{aligned}\right\} \text{ as } r \to \infty. \qquad \text{...(2.4)}$$

We expand the streamfunction Ψ and Ω as

$$\Psi = \Psi_0 + Re\,\Psi_1 + \dots \qquad \text{...(2.5)}$$

$$\Omega = \Omega_0 + Re\,\Omega_1 + \dots \qquad \text{...(2.6)}$$

We substitute (2.5) and (2.6) in (2.1) and (2.2) and equating like powers of Re on both sides we get for terms free of Re.

$$D^4\,\Psi_0 = 0, \quad D^2\,\Omega_0 = 0 \qquad \text{...(2.7)}$$

with $B.\ C$'s

$$\Omega_0 = \Psi_0 = \frac{\partial\Psi_0}{\partial r} = 0 \text{ on } r = 1 \qquad \text{...(2.8)}$$

$$\Psi_0 = \tfrac{1}{2}\, r^2 \sin^2\theta \qquad \text{as } r \to \infty \qquad \text{...(2.9)}$$

$$\Omega_0 = \tfrac{1}{2}\, c r^2 \sin^2\theta.$$

The solution to the above system of equations is

$$\Psi_0 = \tfrac{1}{2}\left(r^2 - \frac{3}{2} r + \frac{1}{2r} \right) \sin^2\theta \qquad \text{...(2.10)}$$

$$\Omega_0 = \frac{c}{2}\left(r^2 - \frac{1}{r} \right) \sin^2\theta. \qquad \text{...(2.11)}$$

Using Ψ_0 and Ω_0 from (2.10) and (2.11), the governing equations in Ψ_1 and Ω_1 are,

$$D^4\,\Psi_1 = \frac{3c^2}{2r^2} \sin^2\theta \cos\theta \qquad \text{...(2.12)}$$

$$D^2\,\Omega_1 = -\frac{3c}{4}\left(1 - \frac{1}{r^2} \right) \sin^2\theta \cos\theta \qquad \text{...(2.13)}$$

with $b.c$'s

$$\Omega_1 = \Psi_1 = \frac{\partial\Psi_1}{\partial r} = 0 \text{ on } r = 1 \qquad \text{...(2.14)}$$

$$\left.\begin{aligned} \Psi_1 &= 0 \\ \Omega_1 &= 0 \end{aligned}\right\} \text{ as } r \to \infty. \qquad \text{...(2.15)}$$

Solution to the above system of equations in (2.12) and (2.13) are

$$\Psi_1 = (A_1 + B_1 r^3 + C_1 r^5 + D_1 \frac{1}{r^2} + \frac{3c^2}{48} r^2) \sin^2\theta \cos\theta \qquad \text{...(2.16)}$$

$$\Omega_1 = \left(A_2 r^3 + B_2 \frac{1}{r^2} - \frac{3}{36}\, c\, (r^2 - r) \right) \sin^2\theta \cos\theta. \qquad \text{...(2.17)}$$

It is not possible for us to satisfy the condition at infinity on Ψ_1 and Ω_1, similar to Whitehead's Paradox. We therefore introduce Stokes variables and obtain inner solution satisfying condition on the body. By straining the coordinates, we introduce Oseen variables and obtain outer solution satisfying condition at infinity. In the over lapping domain we match the inner solution as $r \to \infty$ and outersolution as $Rer = \rho \to 0$ and fix the arbitrary constant.

3. First Term Inner Solution

The non-dimensional variables are taken as the Stoke's variables. Expanding

$$\Psi^{(s)} = \Psi^{(s)}_{00} + Re\,\Psi^{(s)}_{01} + \ldots \qquad \ldots(3.1)$$

$$\Omega^{(s)} = \Omega^{(s)}_{00} + Re\,\Omega^{(s)}_{01} + \ldots \qquad \ldots(3.2)$$

Taking $\Psi = \Psi^{(s)}$ and $\Omega = \Omega^{(s)}$ in (2.1) and (2.2). The governing equations for $\Psi^{(s)}_{00}$ and $\Omega^{(s)}_{00}$ are

$$D^4\, \Psi^{(s)}_{00} = 0, \; D^2\, \Omega^{(s)}_{00} = 0. \qquad \ldots(3.3)$$

Satisfying the $b.c$'s on the body solution to (3.3) are

$$\Psi^{(s)}_{00} = A\left(r^2 - \frac{3}{2}\,r + \frac{1}{2r}\right)\sin^2\theta \qquad \ldots(3.4)$$

$$\Omega^{(s)}_{00} = B\left(r^2 - \frac{1}{r}\right)\sin^2\theta. \qquad \ldots(3.5)$$

4. First Term Outer Solution

We introduce the Oseen variables as $\rho = Rer$, $\Psi^{(0)} = Re^2\,\Psi^{(s)}$ (Proudman and Pearson 1957)

$$D^2 = Re^2\, D^{(0)^2}. \qquad \ldots(4.1)$$

The Governing equations in (2.1) and (2.2) become

$$D^{(0)^4}\Psi^{(0)} = (c/Re)\left(\cos\theta\,\frac{\partial\Omega^{(0)}}{\partial\rho} - \frac{\sin\theta}{\rho}\,\frac{\partial\Omega^{(0)}}{\partial\theta}\right) \qquad \ldots(4.2)$$

$$D^{(0)^2}\Omega^{(0)} = -(c/Re)\left(\cos\theta\,\frac{\partial\Psi^{(0)}}{\partial\rho} - \frac{\sin\theta}{\rho}\,\frac{\partial\Psi^{(0)}}{\partial\theta}\right) \qquad \ldots(4.3)$$

where $D^{(0)^2} = \dfrac{\partial^2}{\partial\rho^2} + \dfrac{\sin\theta}{\rho^2}\,\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\,\dfrac{\partial}{\partial\theta}\right)$.

Expanding $\Psi^{(0)}$ and $\Omega^{(0)}$ as

$$\Psi^{(0)} = \Psi^{(0)}_{00} + (c/Re)\,\Psi^{(0)}_{01} + \ldots \qquad \ldots(4.4)$$

$$\Omega^{(0)} = \Omega^{(0)}_{00} + (c/Re)\,\Omega^{(0)}_{01} + \ldots. \qquad \ldots(4.5)$$

Governing equations for $\Psi^{(0)}_{00}$ and $\Omega^{(0)}_{00}$ are

$$D^{(0)^4}\,\Psi^{(0)}_{00} = 0, \; D^{(0)^2}\,\Omega^{(0)}_{00} = 0. \qquad \ldots(4.6)$$

The condition at infinity satisfies (4.6). The solutions are

$$\Psi^{(0)}_{00} = \tfrac{1}{2}\rho^2 \sin^2\theta,\quad \Omega^{(0)}_{00} = \frac{c}{2}\rho^2 \sin^2\theta. \qquad ...(4.7)$$

Matching (3.4) and (3.5) as $r \to \infty$ and (4.7) as $\rho \to 0$, the constants A and B are,

$$A = \tfrac{1}{2},\ B = \tfrac{1}{2}c. \qquad ...(4.8)$$

The first term inner solutions are

$$\Psi^{(s)}_{00} = \tfrac{1}{2}\left(r^2 - \frac{3}{2}r + (2r)^{-1}\right)\sin^2\theta,\quad \Omega^{(s)}_{00} = \frac{c}{2}\left(r^2 - r^{-1}\right)\sin^2\theta. \qquad ...(4.9)$$

5. Second Term Inner Solution

Taking (4.9) the governing equation for $\Psi^{(s)}_{01}$ and $\Omega^{(s)}_{01}$ are

$$D^4\,\Psi^{(s)}_{01} = \frac{3c^2}{2r^2}\sin^2\theta\cos\theta \qquad ...(5.1)$$

$$D^2\,\Omega^{(s)}_{01} = -\frac{3c}{4}\left(1 - \frac{1}{r^2}\right)\sin^2\theta\cos\theta. \qquad ...(5.2)$$

Solution after satisfying condition on the body are

$$\Psi^{(s)}_{01} = c^2\ C\left(-\frac{21}{8} + r^3 + \frac{25}{16}\frac{1}{r^2} + \frac{1}{16}r^2\right)\sin^2\theta\cos\theta \qquad ...(5.3)$$

$$\Omega^{(s)}_{01} = cD\left(r^3 - \frac{1}{r^2} - \frac{1}{12}(r^2-1)\right)\sin^2\theta\cos\theta. \qquad ...(5.4)$$

6. Second Term Outer Solution

Taking (4.7) the governing equations for $\Psi^{(0)}_{01}$ and $\Omega^{(0)}_{01}$ are

$$D^{(0)^4}\,\Psi^{(u)}_{01} = 0,\quad D^{(0)^2}\,\Omega^{(0)}_{01} = 0 \qquad ...(6.1)$$

Solution to (6.1) are

$$\Psi^{(0)}_{01} = \rho^3\sin^2\theta\cos\theta \qquad ...(6.2)$$

$$\Omega^{(0)}_{01} = \rho^3\sin^2\theta\cos\theta\,.$$

Matching (5.3) and (5.4) as $r \to \infty$ and (6.2) as $\rho \to 0$, we get $C = D = 1$.

We have the second term inner solutions are

$$\Psi^{(s)}_{91} = c^2\left(-\frac{21}{8} + r^3 + \frac{25}{16r^2} + \frac{1}{16}r^2\right)\sin^2\theta\cos\theta \qquad ...(6.3)$$

$$\Omega^{(s)}_{01} = c\left(r^3 - \frac{1}{r^-} - \frac{1}{12}(r^2-1)\right)\sin^2\theta\cos\theta. \qquad ...(6.4)$$

7. Expression for the Drag

Taking $\Psi = \Psi^{(s)}_{00} + Re\ \Psi^{(s)}_{01}$ and $\Omega = \Omega^{(s)}_{00} + Re\ \Omega^{(s)}_{01}$, the velocity components v_r, v_θ and v_ϕ are

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta} = (1-\frac{3}{2r}+\frac{1}{2r^3})\cos\theta$$
$$+c^2 Re\left(-\frac{21}{8r^2}+\frac{25}{16r^4}+\frac{1}{16}\right)(2\cos^2\theta-\sin^2\theta) \quad \ldots(7.1)$$

$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} = (-1+\frac{3}{4r}+\frac{1}{4r^3})\sin\theta$$
$$+c^2 Re\,(r^2-\frac{1}{r^3}-\frac{1}{12}(r-\frac{1}{r}))\sin\theta\cos\theta \quad \ldots(7.2)$$

$$v_\phi = \frac{\Omega}{r\sin\theta} = \frac{c}{2}(r-\frac{1}{r^2})\sin\theta + cRe\,(r^2-\frac{1}{r^3}-\frac{1}{12}(r-\frac{1}{r})\sin\theta\cos\theta. \quad \ldots(7.3)$$

The expression for the drag is

$$D = \frac{DS}{3}\int_0^\pi [-p+2\frac{\partial v_r}{\partial r})\cos\theta-(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta})\sin\theta]\,r^2\sin\theta\,d\theta \quad \text{on } r=1 \quad \ldots(7.4)$$

where $D_S = 6\pi\mu aU$ and

$$-p = -\int [\frac{\partial^2}{\partial r^2}(rv_\theta)]\,d\theta \text{ on } r=1.$$

The drag D is found to be

$$D = D_S\,(1+\frac{25}{9}c^2\,Re)\,. \quad \ldots(7.5)$$

8. Vorticity

The vorticity components ξ, η, ζ are calculated using Ψ and Ω they are found to be

$$\xi = \frac{1}{r^2\sin\theta}\frac{\partial\Omega}{\partial\theta} = c\,(1-\frac{1}{r^3})\cos\theta+c\,Re\,(r-\frac{1}{r^4}-\frac{1}{12}(1-\frac{1}{r^2}))(2\cos^2\theta-\sin^2\theta) \quad \ldots(8.1)$$

$$\eta = \frac{1}{r\sin\theta}\frac{\partial\Omega}{\partial r} = \frac{c}{2}(2+\frac{1}{r^3})\sin\theta+c\,Re\,(3r+\frac{2}{r^4}-\frac{1}{6})\sin\theta\cos\theta \quad \ldots(8.2)$$

$$\zeta = \frac{1}{r\sin\theta}D^2\Psi = (\frac{1}{r}-\frac{2}{r^3}+\frac{1}{r^4})\sin\theta+c^2\,Re\,(-\frac{1}{4r}+\frac{63}{4r^3})\sin\theta\cos\theta. \quad \ldots(8.3)$$

9. Discussion of the Results

It is really interesting to see that even in a linear equation the iteration on Stokes solution fail just as the well known Whitehead's Paradox. Method of matched asymptotic expansions as developed by Proudman and Pearson (1957) are applied and solution is given for $Re \ll 1$ and $cRe \ll 1$. Correction to Stoke's drag due to rotation is (25/9) $D_S . c^2\,Re$. Due to swirl component v_ϕ, which is zero in the non-rotating case gives two more components of vorticity ξ and η. We find that Ψ and ξ are symmetric about $\theta = 0$ and η, ζ are not symmetric. We know that diffusion of vorticity at sufficiently large distances is important for high Reynolds number flow but not so important for the present results.

References

Kaplun, S., and Lagerstrom, P.A. (1957). Asymptotic expansions of Navier-Stokes solutions for small Reynolds number. *J. Math. Mech.*, **6**, 585–93.

Langlois, W. E. (1964). Slow Viscous Flow. The Macmillan Company, New York.

Proudman, I., and Pearson, J.R.A. (1957). Expansions at small Reynolds number for the flow past a sphere and circular cylinder. *J. Fluid Mech.*, **2**, 237–62.

Oseen, C. W. (1914). Uber die Stokessche Formel Und Uber die verwandi Aufgabe in der Hydrodynamik. *Arkiv Math. astro. Fys.*, 6, N. 29.