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ON GENERALIZED ALPHA-CLOSE-TO-CONVEX FUNCTIONS*

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Let P_a^α denote the class of functions f analytic in the unit disc E with $f(0) = 0$, $f'(0) = 1$ satisfying the condition $\int_{\theta_1}^{\theta_2} \text{Re}\,(J(a, \alpha, f(z))\, d\theta > -\pi$, where

$$J(a, \alpha, f(z)) = \alpha(a + 1)\left\{\frac{(K_{a+2}* f)(z)}{(K_{a+1}* f)(z)} - \tfrac{1}{2}\right\} + (1 - \alpha)\, a\left\{\frac{(K_{a+1} * f)(z)}{(K_a * f)(z)} - \frac{a-1}{2a}\right\},$$

$0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, $z = re^{i\theta}$ $r < 1$, $a > 0$, $\alpha \geqslant 0$, $(K_{a+1} * f)(z)/z \neq 0$ and $(K_a * f)(z)/z \neq 0$ for $z \in E$. In this paper a few properties of the class P_a^α are investigated.

INTRODUCTION

Let A denote the class of functions f analytic in the unit disc E with $f(0) = 0$ and $f'(0) = 1$.

Let $f, g \in A$ where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then $f * g$ where

$$(f * g)(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

is called the Hadamard product or convolution of f and g.

Let $K_a(z) = z(1 - z)^{-a}$, $a \in C$ with Re $a > 0$, where we have chosen a suitable branch so that $K_a \in A$. Let $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, with the properties that $a_n \neq 0$ for all n and $\lim_{n\to\infty} |a_n|^{1/n} = 1$. Then we denote by f^{-1} the unique well defined function f in A for which $f^{-1} * f = K_1$. Further let

$$f_{a,b,c} = K^{-1} * ((1 - c)\, K_{b+1} + cK_b)$$

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where $b, c \in C$ with $\operatorname{Re} b > 0$ and $|c| \leqslant 1$.

Definition 1 (Jankovics 1980) — Let R_a^α denote the class of functions $f \in A$ such that

$$\operatorname{Re}(J(a, \alpha, f(z))) > 0,\ z \in E$$

where
$$J(a, \alpha, f(z)) = \alpha(a+1)\left\{\frac{(K_{a+2} * f)(z)}{(K_{a+1} * f)(z)} - \tfrac{1}{2}\right\} + (1-\alpha)\, a \left\{\frac{(K_{a+1} * f)(z)}{(K_a * f)(z)} - \frac{a-1}{2a}\right\},$$

$a > 0$, $\alpha \geqslant 0$, $(K_{a+1} * f)(z)/z \neq 0$ and $(K_a * f)(z)/z \neq 0$ for $z \in E$.

For $a = 1$, the class R_1^α coincides with the class of α-convex functions. In particular R_1^0 and R_1^1 are the classes of starlike and convex functions respectively. Denoting by S_β the class of all starlike functions of order β, $\beta \leqslant 1$, one can easily verify that

(i) $f \in R_a^0 \Leftrightarrow K_a * f \in S_{(1-a)/2}$

(ii) $f \in R_a^1 \Leftrightarrow K_{a+1} * f \in S_{(1-a)/2}$

and

(iii) $f \in R_a^1 \Leftrightarrow f_{a,a,0} * f \in R_a^0$.

Definition 2 (Jankovics 1980) — Let C_a denote the class of functions $f \in A$ such that

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} J(a, 1, f(z))\, d\theta > -\pi/(a+1)$$

where $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, $z = re^{i\theta}$, $r < 1$ and $a > 0$.

For $a = 1$, the class C_1 reduces to the normalised class of close-to-convex functions. Further it is proved that $f \in C_a$ if and only if there exists a function $g \in R_a^1$ such that

$$\operatorname{Re} \frac{(K_{a+1} * f)(z)}{(K_{a+1} * g)(z)} > 0,\ z \in E.$$

For $a = 1$, this reduces to a well-known result due to Kaplan (1952)

Definition 3 (Bharati 1979) — Let $P(\alpha)$ denote the class of functions $f \in A$, with $f(z) f'(z)/z \neq 0$ for $z \in E$, such that

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right\} d\theta > -\pi,$$

whenever $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, $z = re^{i\theta}$, $r < 1$, α being a non-negative real number.

A function $f \in P(\alpha)$ is called an α-close-to-convex function. The class $P(\alpha)$ unifies the classes of close-to-starlike ($\alpha = 0$) and close-to-convex ($\alpha = 1$) functions.

Theorem A (Bharati 1979) — A function $f \in P(\alpha)$ if and only if there exists a starlike function g in E such that

$$\text{Re}\,\{z^{\alpha} f'^{\alpha}(z)\, f^{1-\alpha}(z)/g(z)\} > 0, \; z \in E.$$

This condition yields immediately a representation formula for functions in $P(\alpha)$ as

$$f(z) = \left\{\frac{1}{\alpha}\int_0^z g^{1/\alpha}(t)\, p^{1/\alpha}(t)\, t^{-1}\, dt\right\}^{\alpha}, \; z \in E$$

where g is starlike and p has positive real part in E.

Theorem B (Bharati 1979) — Let $f \in P(\alpha)$, $\alpha > 0$. Then f is α-convex in $|z| < 2 - \sqrt{3}$. The bound is sharp.

We now introduce the concept of generalized α-close-to-convexity.

Definition 4 — Let P_a^{α} denote the class of functions $f \in A$ such that

$$\int_{\theta_1}^{\theta_2} \text{Re}\, J(a, \alpha, f(z))\, d\theta > -\pi \qquad \ldots(1)$$

whenever $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, $z = re^{i\theta}$, $r < 1$, $a > 0$ and $\alpha \geqslant 0$.

For $a = 1$, the class P_1^{α} coincides with the class $P(\alpha)$ of α-close-to-convex functions and for $\alpha = 1$, the class P_a^1 coincides with the class C_a.

The condition (1) admits the following geometric interpretation : Let

$$\psi = \frac{a+1}{2}\arg z + \alpha \arg \frac{(K_{a+1} * f)(z)}{z} + (1-\alpha)\arg\frac{(K_a * f)(z)}{z}$$

where each argument is chosen to vary continuously. Then the condition (1) simply means that, ψ either increases as θ increases or else decreases in such a manner that it never drops to a value π radians below a previous value.

In this paper we investigate a few properties of the class P_a^α.

Integral Representation

We shall now obtain an analytic criterion for a function f to belong to the class P_a^α.

Theorem 1 — A function $f \in P_a^\alpha$ if and only if there exists a function $g \in R_a^0$ such that

$$\operatorname{Re} \frac{(K_{a+1} * f)^\alpha (z) (K_a * f)^{1-\alpha}(z)}{(K_a * g)(z)} > 0, z \in E.$$

Proof : Let $f \in P_a^\alpha$ so that $(K_{a+1} * f)(z)/z \neq 0$ and $(K_a * f)(z)/z \neq 0$ for $z \in E$. Choose suitable branches of $\log ((K_{a+1} * f)(z)/z)$ and $\log ((K_a * f)(z)/z)$ and put

$$t(\theta) = \frac{(a+1)\theta}{2} + \alpha \arg \frac{(K_{a+1} * f)(re^{i\theta})}{re^{i\theta}} + (1-\alpha) \arg \frac{(K_a * f)(re^{i\theta})}{re^{i\theta}}.$$

The condition (1) then reduces to

$$t(\theta_2) - t(\theta_1) > -\pi$$

whenever $\theta_2 > \theta_1$. We observe further that

$$t(\theta + 2\pi) = t(\theta) + (a+1)\pi. \text{ Define } s(\theta) = \tfrac{1}{2}\pi + \inf_{\theta' \geqslant \theta} t(\theta').$$

Then, $s(\theta)$ increases with θ, $s(\theta + 2\pi) = s(\theta) + (a+1)\pi$ and

$$| t(\theta) - s(\theta) | \leqslant \pi/2. \qquad \ldots(2)$$

Now set $p(\theta) = \frac{1}{a+1} s(\theta) - \frac{\theta}{2}$. Then $p(\theta)$ is periodic with period 2π.

We can therefore find an analytic function q_r such that

$$| q_r'(0) | = | f'(0) |, \quad \arg (q_r(re^{i\theta})/re^{i\theta}) = p(\theta) \text{ and}$$

$$\operatorname{Re} (zq_r'(z)/q_r(z)) > 1/2.$$

The inequality (2) together with maximum principle yields

$$\left| \alpha \arg \frac{(K_{a+1} * f)(\rho e^{i\theta})}{\rho e^{i\theta}} + (1-\alpha) \arg \frac{(K_a * f)(\rho e^{i\theta})}{\rho e^{i\theta}} - (a+1) \arg \frac{q_r(\rho e^{i\theta})}{\rho e^{i\theta}} \right| < \pi/2$$

for $0 \leqslant \rho \leqslant r$. Choose a sequence $\{r_n\}$ of values of r increasing to 1 and apply Montel's theorem to $\{q_{r_n}\}$ to obtain a function $q \in S^*_{1/2}$, for which we have

$$\left| \alpha \arg \frac{(K_{a+1} * f)(z)}{z} + (1-\alpha) \arg \frac{(K_a * f)(z)}{z} - (a+1) \arg \frac{q(z)}{z} \right| < \pi/2.$$

Define a function g so that

$$g(z) = K_a^{-1}(z) * z\,\{q(z)/z\}^{a+1}, z \in E.$$

Then we can easily see that $K_a * g$ is a starlike function of order $(1-a)/2$. That is, $g \in R_a^0$ and $(a+1) \arg (q(z)/z) = \arg ((K_a * g)(z)/z)$. Hence we have

$$\operatorname{Re} \frac{(K_{a+1} * f)^{\alpha}(z)\,(K_a * f)^{1-\alpha}(z)}{(K_a * g)(z)} > 0, z \in E.$$

The proof of the converse part follows easily.

Remark : By the above theorem, for $\alpha = 1, f \in P_a^1$ if and only if

$$\operatorname{Re} \frac{(K_{a+1} * f)(z)}{(K_a * g)(z)} > 0, z \in E,$$

where $g \in R_a^0$. Using the result that

$$G \in R_a^1 \Leftrightarrow f_{a,a,0} * G \in R_a^0$$

the above condition reduces to

$$\operatorname{Re} \frac{(K_{a+1} * f)(z)}{(K_{a+1} * G)(z)} > 0, z \in E$$

where $G \in R_a'$, which is a result established by Jankovics (1980). For $a = 1$ we deduce Theorem A.

The analytic condition obtained above provides an integral representation for functions in the class P_a^{α}. In this connection we have the following:

Theorem 2 — A function f is in P_a^{α}, $\alpha > 0$, $a > 0$ if and only if

$$(K_a * f)(z) = \left\{ \frac{a}{\alpha} z^{(1-a)/\alpha} \int_0^z t^{-1+(a-1)/2\alpha} h^{(a+1)/2\alpha}(t) p^{1/\alpha}(t)\, dt \right\}^{\alpha}, z \in E$$

where h is starlike and p has positive real part in E.

PROOF : A function f is in P_a^{α} if and only if

$$(K_{a+1} * f)^{\alpha}(z)\,(K_a * f)^{1-\alpha}(z) = (K_a * g)(z)\,p(z), \ z \in E, \quad \ldots(3)$$

where $g \in R_a^0$ and p has positive real part in E. Also we know that

$$g \in R_a^0 \Leftrightarrow K_a * g \in S_{(1-a)/2}$$

and

$$H \in S_\beta \Leftrightarrow H(z) = z(h(z)/z)^\beta,\ h \text{ starlike and } \beta \leqslant 1.$$

Taking $H = K_a * g$ and $\beta = (1 - a)/2$, (3) reduces to

$$(K_{a+1} * f)(z)\,(K_a * f)^{1/\alpha - 1}(z) = z^{(1-a)/2\alpha} h^{(a+1)/2\alpha}(z) p^{1/\alpha}(z),$$

which by using the relation

$$\frac{z(K_a * f)'(z)}{(K_a * f)(z)} = \frac{a(K_{a+1} * f)(z)}{(K_a * f)(z)} - (a - 1),$$

becomes

$$z(K_a * f)^{(1/\alpha)-1}(z)\,(K_a * f)'(z) + (a - 1)\,(K_a * f)^{1/\alpha}(z)$$
$$= a z^{(1-a)/2\alpha} h^{(a+1)/2\alpha}(z)\, p^{1/\alpha}(z). \quad \ldots(4)$$

Multiplying both sides of (4) by $\alpha^{-1} z^{-1+(a-1)/\alpha}$, the left-hand side becomes the exact differential of $z^{(a-1)/\alpha} (K_a * f)^{1/\alpha}$. Hence on integrating from 0 to z, we get the required representation formula.

Remark : For $a = 1$, Theorem 2 yields the representation for functions in $P(\alpha)$.

OTHER PROPERTIES OF P_a^α

Theorem 3 — Let $f \in P_a^\alpha$, $\alpha > 0$ and $a > 0$. Then

$$\text{Re } J(a, \alpha, f(z)) > 0,$$

for $|z| < [3 + a - 2\sqrt{a + 2}]/(a + 1)$. The bound is sharp.

PROOF : $f \in P_a^\alpha$ implies that

$$(K_{a+1} * f)^\alpha(z)\,(K_a * f)^{1-\alpha}(z) = (K_a * g)(z)\, p(z) \quad \ldots(5)$$

where $g \in R_a^0$ and p has positive real part in E. This yields,

$$J(a, \alpha, f(z)) = \frac{z(K_a * g)'(z)}{(K_a * g)(z)} + \frac{zp'(z)}{p(z)} - \frac{1 - a}{2}.$$

Since $g \in R_a^0$, $K_a * g \in S_{(1-a)/2}$. Then using the well-known distortion theorems we have, for $|z| = r$

$$\operatorname{Re} J(a, \alpha, f(z)) \geqslant \frac{1-ar}{1+r} - \frac{2r}{1-r^2} - \frac{1-a}{2}$$

$$> 0, \text{ if } |z| < \frac{3+a-2\sqrt{a+2}}{a+1}.$$

Taking $g(z) = K_a^{-1}(z) * z/(1+z)^{a+1}$ and $p(z) = (1-z)/(1+z)$ in (5) we see that $J(a, \alpha, f(z)) = 0$ for $|z| = (3+a-2\sqrt{a+2})/(a+1)$.

Remark : For $a = 1$, Theorem 3 gives the radius of α-convexity for functions in $P(\alpha)$, a result contained in Theorem *B*.

Theorem 4 — If $f \in P_a^\alpha$, $a > 0$, $\alpha > 0$ and $M(r) = \max |(K_a * f)(re^{i\theta})|$, $0 \leqslant \theta \leqslant 2\pi$, then

$$M(r) = O[(1-r)^{\alpha-(a+2)}], \text{ for } 0 \leqslant \alpha < a+2,$$

$$= O[\log(1-r)]^{-\alpha}, \text{ for } \alpha = a+2,$$

as $r \to 1^-$. If $\alpha > a+2$, then we have

$$M(r) < \left\{\frac{a}{\alpha} 2^{1/\alpha} r^{(1-a)/\alpha} \frac{\Gamma(a/\alpha)\Gamma(1-(a+2)/\alpha)}{\Gamma(1-(2/\alpha))}\right\}^\alpha.$$

PROOF : By Theorem 2,

$$(K_a * f)(z) = \left\{\frac{a}{\alpha} z^{(1-a)/\alpha} \int_0^z t^{-1+(a-1)/2\alpha} h^{(a+1)/2\alpha}(t)\, p^{1/\alpha}(t)\, dt\right\}^\alpha$$

where h is starlike and p has positive real part in E. Using the well-known distortion theorems for h and p we have, for $|z| = r$

$$|(K_a * f)^{1/\alpha}(z)| \leqslant \frac{a}{\alpha} r^{(1-a)/\alpha} \int_0^r t^{-1+(a/\alpha)}(1+t)^{1/\alpha}(1-t)^{-(a+2)/\alpha}\, dt$$

$$\leqslant \frac{a}{\alpha} 2^{1/\alpha} r^{(1-a)/\alpha} \int_0^r t^{-1+(a/\alpha)}(1-t)^{-(a+2)/\alpha}\, dt$$

$$= \frac{a}{2} 2^{1/\alpha} r^{1/\alpha} \int_0^1 u^{-1+(a/\alpha)}(1-ru)^{-(a+2)/\alpha}\, du.$$

Thus we have

$$| (K_a * f)^{1/\alpha}(z) | \leqslant 2^{1/\alpha} r^{1/\alpha} F\left(\frac{a}{\alpha}, \frac{a+2}{\alpha}, \frac{a}{\alpha}+1; r\right) \qquad \ldots(6)$$

where F is the hypergeometric function. If we now make the restriction $0 < \alpha < a + 2$, then

$$\lim_{r \to 1^-} \frac{F\left(\frac{a}{\alpha}, \frac{a+2}{\alpha}, \frac{a}{\alpha}+1; r\right)}{(1-r)^{1-(a+2)/\alpha}} = \frac{a}{a+2-\alpha}$$

and if we combine this with (6) we obtain, as $r \to 1^-$,

$$M(r) = O[(1-r)^{\alpha-(a+2)}].$$

If $\alpha = a + 2$, we have

$$\lim_{r \to 1^-} \frac{F\left(\frac{a}{\alpha}, 1, \frac{a}{\alpha}+1; r\right)}{\log (1-r)^{-1}} = \frac{a}{\alpha},$$

and combining this with (6) we obtain as $r \to 1^-$

$$M(r) = O[\log (1-r)]^{-\alpha}.$$

In the case when $\alpha > a + 2$, we have

$$\int_0^r t^{-1+(a/\alpha)}(1-t)^{-(a+2)/\alpha}\, dt < \int_0^1 t^{-1+(a/\alpha)}(1-t)^{-(a+2)/\alpha}\, dt$$

$$= \frac{\Gamma(a/\alpha)\, \Gamma(1-(a+2)/\alpha)}{\Gamma(1-2/\alpha)}$$

and therefore

$$M(r) < \left\{\frac{a}{\alpha}\, 2^{1/\alpha} r^{(1-a)/\alpha} \frac{\Gamma(a/\alpha)\, \Gamma(1-(a+2)/\alpha)}{\Gamma(1-2/\alpha)}\right\}^{\alpha}.$$

Theorem 5 — Let $f \in P_a^{\alpha}$, $a > 0$, $\alpha > 0$. Then the function F defined by

$$(K_a * F)(z) = (K_a * f)(z) \left\{\frac{(K_{a+1} * f)(z)}{(K_a * f)(z)}\right\}^{\alpha}, \; z \in E$$

belongs to P_a^0.

Theorem 6 — Let $f \in P_a^{\alpha}$, $a > 0$, $\alpha > 0$. Then the function F defined by

$$(K_{a+1} * F)(z) = (K_a * f)(z) \left\{\frac{(K_{a+1} * f)(z)}{(K_a * f)(z)}\right\}^{\alpha}, \; z \in E$$

belongs to P_a^1.

The proofs of Theorem 5 and 6 are straight-forward and therefore omitted.

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