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COHOMOLOGY THEORY OF GRADED LIE ALGEBRAS

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In this communication we define the cohomology groups H^n_ϕ (g, h) where g, h are graded Lie algebras. We show the equivalence H^2_ϕ (g, h) $\simeq$ Opext$^2_\phi$ (g, h) for h abelian and compute H^n_ϕ (g, V) $\forall\, n > 0$, where g is the Dirac algebra with k generators, V is a complex graded vector space and ϕ ([g, g]) $= 1_V$.

§ 1. Let g,h be (Z_2-) graded Lie algebras over $F(=R$ or $C)$ and let ϕ : g $\to$ End h be a representation [Nijenhuis 1966].

Definition 1—An n-form on g with values in h is an F-linear mapping.

f : $\mathbf{g}^n \to \mathbf{h}$ which satisfies

$$f(X_1,\ldots, X_i, X_{i+1}, \ldots X_n) = (-1)^{|X_i|\cdot|X_{i+1}|} f(X_1, \ldots X_{i+1}, X_i, \ldots X_n) \quad \ldots(1)$$

$\forall\, X_k \in$ g, $k = 1, \ldots, n$, $i = 1, \ldots n-1$ and where $|X_i|$ is the Z_2-degree of X_i.
Let C^n (g, h) be the set of all such n-forms.
Define $\partial : C^n$ (g, h) $\to C^{n+1}$ (g, h) by

$$\partial f(X_1, \ldots X_n, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{\sigma_i} (-1)^i X_i, f(X_1, X_2, \ldots \hat{X}_i, \cdots X_{n+1})$$

$$+ \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} (-1)^{i+j} (-1)^{\sigma_{ij}} f([X_i, X_j], X_1, \ldots \hat{X}_i, \ldots \hat{X}_j \cdots X_{n+1}) \quad \ldots(2)$$

where $\sigma_i = |X_i| \sum_{j=1}^{i-1} |X_j|$

$$\sigma_{ij} = \sigma_i + \sigma_j + |X_i| . |X_j|$$

$$X . f(X_1, \ldots, X_n) \equiv \phi(X) . f(X_1, \ldots X_n)$$

and where the cap '$\wedge$' indicates that the argument is to be omitted.

Proposition 1—$\partial^2 = 0$: Let C° (g, h) $=$ Hom (F, h) $\equiv \{f : F \to h : f$ is linear $\}$, and let

$$\partial f(X) = X. f(1)\ f \in C^\circ (\mathbf{g}, \mathbf{h}),\ X \in \mathbf{g} \quad \ldots(3)$$

Finally, define $C^{-1} \simeq \{0\}$ and

$$H^n_\phi\ (\mathbf{g},\mathbf{h}) = (\text{Ker}\ \partial: [C^n \to C^{n+1}]\ )/\ \partial\ C^{n-1} \qquad \ldots(4)$$

where C^n is short for C^n $(\mathbf{g}, \mathbf{h})$. The (abelian) groups H^n_ϕ $(\mathbf{g}, \mathbf{h})$ have the natural structure of a graded vector space. In contrast to the ungraded case $n >$ dim $\mathbf{g}$ need not imply that H^n_ϕ $(\mathbf{g}, \mathbf{h})$ is trivial.

§ 2. Let $\mathcal{GLA}$ be the category of graded Lie algebras over F with morphisms as graded Lie algebra homomorphisms, i.e. F-linear maps $f : \mathbf{g} \to \mathbf{h}$ which satisfy

(i) $|f(X)| = |X|$

(ii) $f([X, Y]) = [f(X), f(Y)] \quad \forall\ X, Y \in \mathbf{g}$ ...(5)

Assume that $\mathbf{h}$ is graded abelian.

Definition 2 — An extension of $\mathbf{h}$ by $\mathbf{g}$ is a short exact sequence

$$E: \quad O \to \mathbf{h} \xrightarrow{\mu} \mathbf{k} \xrightarrow{\nu} \mathbf{g} \to 0 \qquad \ldots(6)$$

n $\mathcal{GLA}$.

Congruence of extensions and the sum of two extensions are defined as usual [MacLane 1963]. We assume, in the sequel, that the sequence E splits in the category of graded vector spaces. Now, the adjoint action in $\mathbf{k}$ leads to the mapping $\theta : \mathbf{k} \to$ End $\mathbf{h}$, where

$$\mu\ ((\theta\ X)\ .\ Y) = [\ X, \mu\ (Y)\ ],\ X \in \mathbf{k},\ Y \in \mathbf{h}. \qquad \ldots(7)$$

Since $\mathbf{h}$ is graded abelian, $\theta.\ \mu = 0$ and hence θ induces a homomorphism $\phi : \mathbf{g} \to$ End $\mathbf{h}$ such that the diagram

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{\nu} & \mathbf{g} \\ \theta \downarrow & \swarrow \phi & \\ \text{End}\ \mathbf{h} & & \end{array} \qquad \ldots(8)$$

commutes [MacLane 1963, p. 252]. We then call E as an extension of $\mathbf{h}$ by $\mathbf{g}$ with operators ϕ. Note of course that ϕ makes $\mathbf{h}$ a $\mathbf{g}$ module. Let Opext$^2_\phi$ $(\mathbf{g},\mathbf{h})$ denote the group of all such sequences (modulo congruences).

Theorem 1 — H^2_ϕ $(g, h) \simeq$ Opext$^2_\phi$ $(\mathbf{g}, \mathbf{h})$ as sets

PROOF : Choose a mapping $u : \mathbf{g} \to \mathbf{k} : \nu.\ u = 1_{\mathbf{g}}$. This is possible since E splits in the category of graded vector spaces; u is then a morphism in this category. Identify $\mu\ (Y)$ with $Y\ \forall\ Y \in \mathbf{h}$ and denote $\phi\ (X).\ Y \equiv X\ .\ Y,\ \forall\ X \in g,\ Y \in \mathbf{h}$

$$\because [u\ (X), \mu\ (Y)] = [u\ (X),\ Y\ ] = \mu\ (\ (\theta\ .\ u\ (X)\ )\ .\ Y)$$
$$= \mu\ (\phi\ (X)\ .\ Y) = \phi\ (X)\ .\ Y = X\ .\ Y\ \forall\ X \in \mathbf{g}\ Y \in \mathbf{h}. \qquad \ldots(9)$$

Thus, $X.\ Y = [u\ (X),\ Y],\ X \in \mathbf{g},\ Y \in \mathbf{h}$.

Now $[u\ (X),\ u\ (Y)]$ must lie in the same coset in $\mathbf{k}/\mathbf{h}$ as

$u\ ([X, Y])\ \forall\ X,\ Y \in \mathbf{g}$. Therefore,

$$\exists\ f : \mathbf{g}^2 \to \mathbf{h} : [u\ (X), u\ (Y)] = u\ ([X, Y]) + f\ (X, Y)\ \forall\ X,\ Y \in \mathbf{g}. \qquad \ldots(10)$$

It is trivial to verify that $f \in C^2(\mathbf{g}, \mathbf{h})$. Further, the Jacobi condition

$$(-1)^{|X| \cdot |Z|} [u(X), [u(Y), u(Z)]] + \text{cyclic terms} = 0 \qquad ...(11)$$

leads to the condition $\partial f = 0$. Finally the choice $u'(X) = u(X) + V(X)$, $V : \mathbf{g} \to \mathbf{h}$ leads to

$$f'(X, Y) = f(X, Y) + \partial v(X, Y). \qquad ...(12)$$

It is easy to show that the extensions defined by f, f' are equivalent. Hence the theorem.

§3. Let g be the Dirac algebra over k generators $\{J_0, J_1, \dots J_k\}$ with

(i) $|J_0| = 0$, $|J_i| = 1$

(ii) $[J_i, J_j] = 2\delta_{ij} J_0$, $i, j = 1, \dots, k$ all other independent brackets being zero.

Let V be a complex vector space and let ϕ be a representation of g on V such that

$$\phi([\mathbf{g}, \mathbf{g}]) = \phi(J_0) = 1_V.$$

Then one has

Theorem 2 — $H^n_\phi(\mathbf{g}, V) = \{0\} \; \forall \; n > 0$.

PROOF: Let $f \in C^n(\mathbf{g}, V)$, $n > 1$. Then

$$\partial f(J_{\mu_1}, J_{\mu_2}, \dots J_{\mu_{n+1}}) = \sum_{i=1}^{n+1} J_{\mu_i} \cdot f(J_{\mu_i}, \dots \hat{J}_{\mu_i}, \dots J_{\mu_{n+1}}) - 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \delta_{\mu_i \mu_j}$$

$$\times f(J_0, J_{\mu_1}, \dots \hat{J}_{\mu_i}, \dots \hat{J}_{\mu_j} \dots J_{\mu_{n+1}}) \qquad ...(13)$$

and $$\partial f(J_0, J_{\mu_1}, \dots J_{\mu_n}) = f(J_{\mu_1}, \dots, J_{\mu_n}) - \left(\sum_{i=1}^{n} J_{\mu_i} f(J_0, J_{\mu_1}, \dots J_{\mu_i}, \dots \hat{J}_{\mu_n})\right)$$

$$\forall \; J_{\mu_i} \in \mathbf{g}/\{J_0\}. \qquad ...(14)$$

It is easy to see that if the R. H. S. of eqn. (14) is zero, then so is the R. H. S. of eqn. (13). Thus, $\partial f = 0 \Leftrightarrow$

$$f(J_{\mu_1}, \dots J_{\mu_n}) = \Sigma J_{\mu_i} f(J_0, \dots \hat{J}_{\mu_i}, \dots J_{\mu_n}) \; \forall \; J_{\mu_i} \in \mathbf{g} \{\hat{J}_0\} \qquad ...(15)$$

Let $f' = f + \partial\omega$, $\omega \in C^{n-1}(\mathbf{g}, V)$ and choose ω in such a way that $f'(J_0, \dots \hat{J}_{\mu i} \dots J_{\mu_u}) = 0$, i.e., such that $f(J_0, J_{\mu_1}, \dots \hat{J}_{\mu_i} \dots J_{\mu n}) + \omega(J_{\mu 0}, \dots \hat{J}_{\mu_i}, J_{\mu n})$

$$-\left(\sum_{j \neq i} J_{\mu j} \omega(J_0, \dots \hat{J}_{\mu i}, \dots \hat{J}_{\mu j}, J_{\mu_n}\right) = 0. \qquad ...(16)$$

Note that this can always be done; for example

$$\omega(J_{\mu 1}, \dots \hat{J}_{\mu_i}, \dots J_{\mu n}) = -f(J_0, \dots J_{\mu i}, \dots \hat{J}_{\mu i}, \dots J_{\mu n})$$

$$\omega(J_0, \dots \hat{J}_{\mu i}, \dots \hat{J}_{\mu j} \dots J_{\mu n}) = 0$$

is a solution. Now, we know that $\partial f' = 0$; eqn. (15) yields immediately that $f' = 0$. Hence $H^n(\mathrm{g}, V) = \{0\} \; \forall \; n > 1$. For the case $n = 1$, let $f \in C^1(g, V)$. Then, $\partial f = 0 \Leftrightarrow$

$$f(J) = J.f(J_0) \; \forall \; J \in \mathrm{g}/\{J_0\}. \qquad \text{...(18)}$$

Define $f' = f + \partial\omega$ and choose ω such that $f'(J_0) = 0$ i.e. $f(J_0) + \omega(1) = 0$. One readily has $f' = 0$, implying that $H^1_\phi(\mathrm{g}, V) = \{0\}$.

References

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