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ON SOME FIXED POINT THEOREMS IN UNIFORM SPACES

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In this paper we extend a recent fixed point result of Browder for contractive type mappings on metric spaces to uniform spaces and obtain in the same setting a common fixed point result for a pair of mappings. This yields as corollaries certain known results of Boyd and Wong (1969) and Rhoades (1977).

1. INTRODUCTION

Starting with the well-known contraction principle of Banach, there has been an abundant growth of literature on fixed point theorems of contractive-type in the setting of metric spaces. An excellent survey of the related results has been given recently in Rhoades (1977), wherein 250 different definitions of mappings of contractive-type are compared and some of the most up-to-date results in this direction are given. Very recently, Browder (1979) has formulated in the same spirit a general contraction principle which subassumes many known results of this type.

Extensions of the Banach's contraction theorem and some of the related results to the framework of uniform spaces is known for quite some time. (e.g. cf. Tarafdar 1974, Cain and Nashed (1971), Pai and Veeramani 1980 etc.). In this paper we extend the principal fixed point result of Browder (1979) to uniform spaces (Theorem 2.4). We remark that by employing similar techniques almost all the results of Rhoades (1977), extend to uniform spaces. We also obtain a general fixed point result for the common fixed points of a pair of mappings (Proposition 3.1). This enables us to combine and extend a result of Boyd and Wong (1969, Theorem 2) and a result of Rhoades (1977, Theorem 14) to uniform spaces (Theorem 3.2).

2. A GENERAL FIXED POINT PRINCIPLE

Let $(X, \mathscr{U})$ be a uniform space. For the terminology of uniform spaces, we refer the reader to Thron (1966). Let $\mathscr{D} = \{d_i : i \in I\}$ denote a family of uniformly continuous pseudometrics on X that generates $\mathscr{U}$. We readily observe that for $i_1, i_2 \in I$, max $\{d_{i_1}, d_{i_2}\}$ is again a uniformly continuous pseudometric on X. Thus augmenting $\mathscr{D}$ if necessary, by adding maxima of finitely many members of $\mathscr{D}$ we may assume, without loss of generality, that the family $\{B_i(\epsilon) : i \in I, \epsilon > 0\}$, where $B_i(\epsilon) = \{(x, y) : d_i(x, y) < \epsilon\}$ constitutes a base for the uniformity $\mathscr{U}$. Such a family $\mathscr{D}$ is called an 'augmented associated family of pseudometrics' for $\mathscr{U}$. It is

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well-known that for each uniformity $\mathscr{U}$ on X, there exists an augmented associated family $\mathscr{D}$ of pseudometrics (cf. Thron 1966, p. 177).

Let R^+ denote the nonnegative reals and let Φ denote an indexed family $\{\varphi_i\}i \in I$ of functions φ_i ; $R^+ \longrightarrow R^+$ that are non-decreasing, upper semi-continuous from the right and, satisfying $\varphi_i(t) < t$ if $t > 0$. In the framework of metric spaces, functions of this type have been employed for studying non-linear contractions by Boyd and Wong (1969) and more recently by Browder (1979). In Browder (1979) these have been called contractive gauge functions. The indexing set I for Φ is to be understood as the same as that for $\mathscr{D}$. Given a set $E \subset X$, we shall denote by $\delta_i(E)$ the d_i-diameter of E, i.e., $\delta_i(E) = \sup \{d_i (x, y) : x, y \in E\}$.

Lemma 2.1—Let $\varphi \in \Phi$ and $t_0 > 0$. Then the sequence $\varphi^n (t_0) \longrightarrow 0$ as $n \longrightarrow \infty$.

PROOF : This follows immediately from the stated properties of φ.

Lemma 2.2—Let X be a sequentially complete Hausdorff uniform space and let $\{E_n\}$ be a sequence of nested subsets of X, $E_{n+1} \subset E_n$ for each $n \in \mathbb{N}$. For each $i \in I$, let $\delta_i^{(n)} = \delta_i (E_n)$ and assume that $\delta_i^{(n+1)} \leqslant \varphi_i (\delta_i^{(n)})$ for each $i \in I$. Then $E = \cap \{\text{cl } E_n : n \in \mathbb{N}\}$ consists of a single point.

PROOF : By Lemma 2.1, for each $i \in I$, $\delta_i^{(n)} \longrightarrow 0$ as $n \longrightarrow \infty$. Select $x_n \in E_n$ arbitrarily for $n \in \mathbb{N}$. Then $d_i (x_n, x_m) \longrightarrow 0$ as $m, n \longrightarrow \infty$ for each $i \in I$ and hence $\{x_n\}$ is a cauchy sequence in X. Since X is sequentially complete, $x_n \longrightarrow x_0 \in X$. Clearly $x_0 \in E$. Also since $\delta_i (E) \leqslant \varphi_i^{(n)}$ for each n, $\delta_i (E) = 0$ for each $i \in I$, whence $E = \{x_0\}$.

Proposition 2.3—Let X be a sequentially complete Hausdorff-uniform space. Let X_0 be a subset of X and let T be a continuous self-mapping of X which carries X_0 into X_0. For any pair $\{x, y\}$ in X, let $0 (T, x, y)$ denote the orbit of the pair under f, i. e., $0 (T, x, y) = \underset{n \geqslant 0}{U} [T^n x] \cup \{T^n y\}$. Let $\Phi = \{\varphi_i : i \in I\}$ be a given family of functions as in Lemma 2.2. Suppose that for each $x \in X_0$ and $i \in I$, there is a $p_i (x) \in \mathbb{N}$ such that for any pair x, y in X_0 there holds

$$\delta_i (0\, T, T^{p_i(x)+p_i(y)} (x),\ T^{p_i(x)+p_i(y)} (y)) \leqslant \varphi_i (\delta_i (0 (T, x, y)), \text{ for each } i \in I.$$

Then for each $x \in X_0$, $T^n x$ converges to a fixed point of T in X.

PROOF : Let $A_n = 0 (T, T^n x, T^n y)$. Clearly $A_{n+1} \subset A_n$. Fix up $i \in I$. Choose a sequence $k_i (n)$ of positive integers by iteration such that if $k_i (n)$ is given, then with $x_n = T^{k_i(n)} (x)$, $y_n = T^{k_i(n)} (y)$, we set $k_i (n + 1) = k_i (n) + p_i (x_n) + p_i (y_n)$. If we denote by $E_n^{(i)}$ the set $A_{k_i} (n)$, it follows from the hypothesis of the theorem that $\delta_i (E_{n+1}^{(i)}) \leqslant \varphi_i (\delta_i (E_n^{(i)}))$. Hence by Lemma 2.1, $\delta_i (E_n^{(i)}) \longrightarrow 0$.

Since $\delta_i (A_n)$ is a non-increasing sequence of R^+, $\delta_i (A_n) \to 0$ as its infinite subsequence $\delta_i (A_{ki(n)})$ converges to zero.

By Lemma 2.2, the set $A = \cap \{\text{cl } (A)_n : n \in \mathbb{N}\}$ consists of a single point x_0. Since $T (A_n) \subset A_{n+1}$ and T is continuous, $T (\text{cl } A_n) \subset \text{cl } A_{n+1}$, whence $T(A) \subset A$. Since A consists of a single point x_0, x_0 is a fixed point of T.

Theorem 2.4—Let X be a sequentially complete Hausdorff uniform space. Let X_0 be a subset of X and T be a continuous self-mapping of X which carries X_0 into

X_0. Let $\Phi = \{\varphi_i : i \in I\}$ be as in Lemma 2.2. For each $x \in X_0$ and $i \in I$ suppose that there is $p_i(x) \in \mathbb{N}$ such that for $n \geqslant p_i(x)$ and $y \in X_0$ there holds

$$d_i(T^n x, T^n y) \leqslant \varphi_i(\max\{\sup_{(p,q)\in I_1} d_i(T^p x, T^q y), \sup_{(r,s)\in I_2} d_i(T^r x, T^s x), \sup_{(l,m)\in I_3} d_i(T^l y, T^m y)\})$$

for each $i \in I$, I_1, I_2, I_3 being arbitrary subsets of $\mathbb{N}\times\mathbb{N}$. Then T has a fixed point x_0 in X and for each $x \in X_0$, $T^n x \longrightarrow x_0$ as $n \longrightarrow \infty$.

PROOF : By proposition 2.3, it would suffice to show that under the given hypothesis there holds :

$$\delta_i(0(T, T^{p_i(x)+p_i(y)}(x), T^{p_i(x)+p_i(y)}(y))) \leqslant \varphi_i(\delta_i\, 0(T;x,y))).$$

Let $z, w \in 0(T; T^{p_i(x)+p_i(y)}(x), T^{p_i(x)+p_i(y)}(y))$.

Fix up $i \in I$. After interchanging z, w or x, y if necessary, we may assume that $z = T^n x$, $w = T^{n+t} y$ for $n \geqslant p_i$

$$\text{Then } d_i(z, w) = \varphi_i(T^n x, T^n(T^t y))$$
$$\leqslant \varphi_i(\max\{\sup_{(p,q)\in I_1} d_i(T^p x, T^{q+t} y)$$
$$\sup_{(r,s)\in I_2} d_i(T^r x, T^s x), \sup_{(l,m)\in I_3} d_i(T^{l+t} y, T^{m+t} y)\})$$
$$\leqslant \varphi_i(\delta_i(0, T, x, y)).$$

The conclusion of the theorem now follows from the previous proposition.

We remark that for metrizable spaces Theorem 2.4 reduces to a recent result of Browder (1979, Theorem 2).

3. COMMON FIXED POINTS OF MAPPINGS

Proposition 3.1—Let X be a sequentially complete Hausdorff uniform space. Let X_0 be a subset of X and let T, S be a pair of continuous self-mappings of X that carry X_0 into X_0. For any $x \in X$, let $0(T, S, x)$ denote the joint orbit of x under T and S, i.e., $0(T, S, x) = \bigcup_{n\geqslant 0} x_n$, where $x_0 = x$, $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$, $n = 0,1,2,\ldots$

Let $\Phi = \{\varphi_i\}$ be a given family of functions as in Lemma 2.2. Suppose that for each $x \in X_0$ and $i \in I$, there is a $p_i(x) \in \mathbb{N}$ such that there holds

$$\delta_i(0(T, S; x_{2p_i}(x))) \leqslant \varphi_i(\delta_i(0(T, S; x))) \text{ for each } i \in I.$$

Then for each $x \in X_0$, there is a common fixed point of T and S in $\bigcap_{n\geqslant 0} \text{cl}(0(T,S,x_{2n}))$, which is obtained as the limit of the sequence $(ST)^n x$.

PROOF : Let $E_n = 0(T, S, x_{2n})$. We readily observe the inclusions $E_0 \supset E_1 \supset E_2 \supset E_3 \ldots$

Let $E = \bigcap_{n\geqslant 0} \text{cl}(E_n)$. Then $E = \bigcap_k \text{cl}(E_{n(k)})$

for any infinite subsequence of integers $n(k)$. Fix up $i \in I$. We choose a sequence $n_i(k)$ literatively by the formula :

$$n_i(k+1) = n_i(k) + p(x_{n2i(k)}).$$

$$\text{Then since } x_{2n_i(k+1)} = (ST)^{n_i(k+1)} x = (ST)^{p(x_{2ni(k)})}(ST)^{n_i(k)}(x)$$
$$= (ST)^{p(x_{2ni(k)})} x_{2n_i(k)} ,$$

it follows from the hypothesis of the proposition that

$$\delta_i (E_{n^i(k+1)}) \leqslant \varphi_i (\delta_i (E_{n^i(k)})).$$

Letting $E_{n_i(k)} = A_k^i$, it follows by Lemma 2.2, that

$\delta_i (A_k^i) \longrightarrow 0$ as $k \longrightarrow \infty$. $\delta_i (E_n)$ being a non-increasing sequence of non-negative reals of which $\delta_i (A_k^i)$ is a subsequence, we obtain that $\delta_i (E_n) \longrightarrow 0$ as $n \longrightarrow \infty$, for each $i \in I$. By Lemma 2.2 again, $E = \bigcap_{n \geqslant 0} \text{cl}\, E_n$ consists of a single point z. Since $x_{2n} = (ST)^n x \in E_n$, $(ST)^n x \longrightarrow z$, as $n \longrightarrow \infty$. Again since $(ST) x_{2n} \longrightarrow z$, the continuity of S and T yields z as a fixed point of ST, i.e., $(ST)\ z = z$. Also since $x_{2n+1} = T(ST)^n x = Tx_{2n}$ and $x_{2n+1} \in E_n$, $x_{2n+1} \longrightarrow z$, whence by continuity of T, we obtain $Tz = z$, $Sz = z$ and z is a common fixed point of S and T.

Theorem 3.2—Let X be a sequentially complete Hausdorff uniform space. Let $\Phi = \{\varphi_i\}$ be a given family of functions as in Lemma 2.2. Let T, S be a pair of self-mappings of X satisfying

$$d_i (Tx, Sy) \leqslant \varphi_i (\max \{d_i (x, y), d_i (x, Tx), d_i (y, Sy), \tfrac{1}{2} [d_i (x, Sy) + d_i (y, Tx)]\})$$

for each $i \in I$ and $x, y \in X$. Then T and S have a unique common fixed point z. Furthermore $(TS)^n x \longrightarrow z$ and $(ST)^n x \longrightarrow z$ for each $x \in X$.

PROOF : Consider the joint orbit $0\,(T, S, x)$ of x under T and S with $x_0 = x$ as in the previous proposition. Assume $x_n \neq x_{n+1}$ for each n. Fix up $i \in I$. Then

$$\begin{aligned} d_i (x_{2n+1}, x_{2n+2}) &= d_i (Tx_{2n}, Sx_{2n+1}) \\ &\leqslant \varphi_i (\max \{d_i (x_{2n}, x_{2n+1}), d_i (x_{2n}, x_{2n+1}), \\ &\quad d_i (x_{2n+1}, x_{2n+2}), \tfrac{1}{2} [d_i (x_{2n}, x_{2n+2}) + 0]\}) \\ &\leqslant \varphi_i (M_i (x_{2n}, x_{2n+2})). \end{aligned}$$

If $M_i (x_{9n}, x_{2n+1}) = d_i (x_{2n}, x_{2n+2})/2$, then $d_i (x_{2n}, x_{2n+1}) \leqslant d_i (x_{2n+1}, x_{2n+2})$

and we obtain the contradiction $d_i (x_{2n+1}, x_{2n+2})$

$$\leqslant \varphi_i (d_i (x_{2n+1}, x_{2n+2})) < d_i (x_{2n+1}, x_{2n+2}).$$

Consequently,

$$\begin{aligned} d_i (x_{2n+1}, x_{2n+2}) &\leqslant \varphi_i (d_i (x_{2n}, x_{2n+1})) \\ &\leqslant \varphi_i^2 (d_i (x_{2n-1}, x_{2n})) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leqslant \varphi_i^{2n} (d_i (x_1, x_2)). \end{aligned}$$

Likewise, $d_i (x_{2n}, x_{2n+1}) \leqslant \varphi_i^{2n} (d_i (x_0, x_1))$

Let $C_i^{(n)} = d_i (x_n, x_{n+1})$. Then $C_i^{(n)} \leqslant \varphi_i^{n-1} (\max \{d_i (x_0, x_1), d_i (x_1, x_2)\})$.

By Lemma 2.1 $C_i^{(n)} \longrightarrow 0$ as $n \longrightarrow \infty$.

We assert that the sequence $\{x_n\}$ is Cauchy. Assume the contrary, then there exist $\epsilon > 0$, $i \in I$ and two sequences $m(j)$ and $n(j)$, $m(j) > n(j) \geqslant j$ for each $j \in \mathbf{N}$ such that

$$d_i^{(j)} = d_i (x_{m(j)}, x_{n(j)}) \geqslant \epsilon, \text{ for } j = 1,2,3, \qquad \ldots(1)$$

We may assume that $d_i(x_{m-1}, x_n) < \epsilon$, by taking m to be the smallest integer $> n$ for which (1) holds.

We have $\epsilon \leqslant d_i^{(j)} \leqslant d_i(x_m, x_{m-1}) + d_i(x_{m-1}, x_n)$

$$\leqslant C_i^{(m-1)} + \epsilon \,. \qquad \ldots(2)$$

Letting $j \longrightarrow \infty$ in (2), we obtain

$d_i^{(j)} \longrightarrow \epsilon_+$ as $j \longrightarrow \infty$.

Also we have $d_i^{(j)} = d_i(x_m, x_n) \leqslant d_i(x_m, x_{m+1}) + d_i(x_{m+1}, x_{n+1}) + d_i(x_{n+1}\, x_n)$

$$\leqslant C_i^{(m)} + \varphi_i(d_i(x_m, x_n)) + C_i^{(n)}$$

$$\leqslant 2C_i^{(k)} + \varphi_i\, d_i^{(j)}). \qquad \ldots(3)$$

Letting $j \longrightarrow \infty$ in (3), we get

$\epsilon \leqslant \varphi_i(\epsilon) < \epsilon$, which is a contradiction for $\epsilon > 0$. Thus $\{x_n\}$ is Cauchy and X being sequentially complete $x_n \longrightarrow z \in X$. We have for each $i \in I$

$$d_i(Tz, z) \leqslant d_i(Tz, x_{2n+2}) + d_i(x_{2n+2}, z) \qquad \ldots(4a)$$

and $d_i(Tz, x_{2n+2}) \leqslant \varphi_i(\max\{d_i(z, x_{2n+1}), d_i(z, Tz), d_i(x_{2n+1}, x_{2n+2}), \frac{1}{2}[d_i(z, x_{2n+2}) + d_i(z, x_{2n+1}, Tz)]\})$. ...(4b)

Since $\max\{d_i(z, x_{2n+1}), d_i(z, Tz), d_i(x_{2n+1}, x_{2n+2}), \frac{1}{2}[d_i(z, x_{2n+2}) + d_i(x_{2n+1}, Tz)]\}$

$\longrightarrow d_i(z_i\, Tz)$ from the right, taking limit as $n \longrightarrow \infty$ in (4), we obtain,

$d_i(Tz, z) \leqslant \varphi_i(d_i(Tz, z)) < d_i(Tz, z)$, whence

$d_i(Tz, z) = 0$ for each $i \in I$ and $Tz = z$. Likewise, $Sz = z$.

Suppose z and w are two fixed points of T and S. Then for each $i \in I$

$d_i(z, w) = d_i(Tz, Sw) \leqslant \varphi_i(\max\{d_i(z, w), 0, 0, \frac{1}{2}[d_i(z,w) + d_i(w,z)]\} < d_i(z,w)$, whence $d_i(z,w) = 0$ and $z = w$.

Remark : The existence of common fixed point of T and S in Theorem 3.2 with the additional hypothesis that T and S be continuous on X also follows from Proposition 3.1. In fact, let $x \in X$. If $\delta_i(0(T, S; x)) = 0$ for each $i \in I$, then clearly x is a common fixed point of T and S. Otherwise, choose ϵ_i such that $0 < \epsilon_j < \varphi_i(\delta_i(0(T, S, x)))$.

As in the proof of Theorem 3.2, $\{x_n\}$ is a Cauchy sequence. Hence, there is $n_i(x) \in \mathbb{N}$ such that $d_i(x_n, x_m) < \epsilon_i$ for all $n, m > n_i(x)$. Let $p_i(x)$ = (even integer next to $n_i(x))/2$). Then evidently.

$\delta_i(0(T, S; x_{2p_i(x)})) \leqslant \varphi_i(\delta_i(0(T, S, x)))$ for each $i \in I$ and by Proposition 3.1, T and S have a common fixed point.

Corollary 3.3—Let X and Φ be as in Theorem 3.2. Let T, S be a pair of self-mappings of X satisfying for each $i \in I$, $d_i(Tx, Sy) \leqslant \varphi_i(d_i(x,y))$ for all $x, y \in X$.

Then T and S have a unique common fixed point which is given as the limit of the sequences $(TS)^n x$, $(ST)^n x$ for each $x \in X$.

In case X is a metric space and $T = S$, Corollary 3.3 is a result of Boyd and Wong (1969, Theorem 2).

Corollary 3.4—Let X be a sequentially complete Hausdorff uniform space. For each $i \in I$, let there be given a constant h_i, $0 \leqslant h_i < 1$. Let T, S be a pair of self-mappings of X satisfying

$$d_i(Tx, Sy) \leqslant h_i \left(\max \{d_i(x, y), d_i(x, Tx), d_i(y, Sy), \tfrac{1}{2}(d_i(x, Sy)\ d_i(y, Tx)]\}\right)$$

for each $i \in I$ and $x, y \in X$. Then T and S have a unique common fixed point which is given as the limit of the sequences $(TS)^n x$, $(TS)^n x$ for each $x \in X$.

In case X is a metric space, Corollary 3.4 is a result of Rhoades (1977, Theorem 14).

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