Indian J. pure appl. Math., 13(6): 682-695, June 1982

GENERATION OF SH WAVES FROM A NONUNIFORMLY MOVING STRESS DISCONTINUITY IN A LAYERED HALF SPACE

J. P. MITTAL*

Department of Physics, Vaish College, Rohtak

AND

R. S. SIDHU

Department of Mathematics, M. D. University, Rohtak

(*Received* 13 *April* 1981)

The generation of SH-type of waves due to nonuniformly moving stress discontinuity in a layered half space is considered. Two different types of the discontinuities are considered. The first type is a concentrated moving discontinuity which is assumed to occur suddenly at the interface and then moves with nonuniform velocity. The second type is a discontinuity which is assumed to occur suddenly at the interface and then expands with nonuniform velocity. The surface displacements have been evaluated using de Hoop's version of Cagniard's technique.

1. INTRODUCTION

Since the classic paper of Lamb (1904) there has been a lot of work on source problems in seismology. In order to interpret the seismic records properly many modifications to the original Lamb problem have been introduced. One such modification, among many others, is to study the response of an elastic medium to moving sources. According to 'elastic rebound' theory, when the elastic limit of the deformed rocks is reached, a fracture occurs which then expands and releases the strain energy of deformation in the form of seismic waves. Moving source problems are intimately connected with the propagation of a fracture in an elastic material. By using the theoretical results for a moving double couple source in a half space, Sudo (1972) was able to interpret a distinct seismic arrival often observed on seismograms of shallow earthquakes. Thus the theoretical study of moving source problems will be of interest in Seismology.

The surface displacement of SH-type of motion due to stress discontinuity moving uniformly along the interface of a layered half space has been studied by Nag (1963). He discussed only the case when the velocity of the source is less than the velocity of S-wave in the upper layer. Transient response of a layered elastic half space subjected to a reciprocating antiplane shear load has been studied by Watanabe (1977). Response of an elastic half space to nonuniformly moving surface loads has been considered by Freund (1973). Roy (1979) has studied the response of an elastic half space to nonuniformly expanding surface loads.

* Department of Mathematics, Vaish College Rohtak.

As nonuniform motion is more likely to occur in nature, we have extended, in this paper, the work of Nag (1963) for nonuniform motion. The technique used is the same as used by Freund (1973). The results for uniform motion can be deduced easily from our results, for all velocities of the source.

2. General Theory

Consider a two layered half space in which the upper layer is a medium 1 of thickness h and the lower layer is a semi-infinite medium 2, occupying the region $z \geqslant 0$. Let β_1 and β_2 be the S-wave velocities in the medium 1 and 2 respectively ($\beta_1 < \beta_2$). The elastic constants and co-ordinate axes are as shown in Fig. 1.

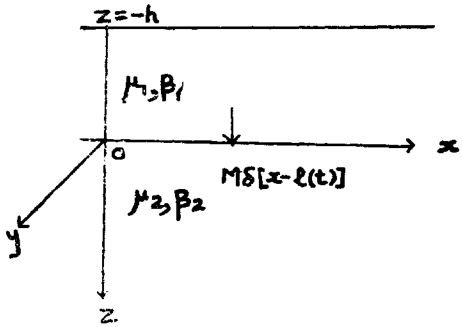

Fig. 1. Layered half space model with a moving stress discontinuity.

Since only SH-type of motion from an infinite line source moving parallel to z-axis is considered, the only non-zero component of displacements are v_1 and v_2 along the y-axis. Thus the only equations of motion to be satisfied are :

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{\beta_1^2} \frac{\partial^2 v_1}{\partial t^2} \qquad \ldots(2.1)$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} = \frac{1}{\beta_2^2} \frac{\partial^2 v_2}{\partial t^2}. \qquad \ldots(2.2)$$

The boundary conditions of the problem are :

(*a*) Continuity of displacement at interface, i.e.

$$v_1 = v_2, \text{ at } z = 0. \qquad \ldots(2.3)$$

(*b*) Vanishing of stress at the free surface, i.e.

$$(\tau_{yz})_1 = \mu_1 \frac{\partial v_1}{\partial z} = 0, \text{ at } z = -h. \qquad \ldots(2.4)$$

(*c*) Shearing stress discontinuity prescribed by,

$$(\tau_{yz})_1 - (\tau_{yz})_2 = \mu_1 \frac{\partial v_1}{\partial z} - \mu_2 \frac{\partial v_2}{\partial z} = f(x, t)\, H(t), \text{ at } z = 0 \qquad \ldots(2.5)$$

where τ_{yz} represents the shearing stress, $H(t)$ is the Heaviside unit function and μ is Lame's constant. Subsequently the following two forms of the function $f(x.t)$ will be considered:

$f(x, t) = M\delta\,[x-l(t)]$ (concentrated moving discontinuity), or

$f(x, t) = P\,[H(x)-H\{x-l(t)\}]$ (expanding discontinuity).

$l(t)$ will be assumed to be a continuous and monotonically increasing function of time, so that $x = l(t)$ is invertible and there exists a continuous and monotonically increasing function $t = \eta(x)$, such that

$$l[\eta(x)] = x, \quad \eta[l(t)] = t. \qquad \ldots(2.6)$$

It will also be assumed that $l(o) = 0$.

By differentiating the equation in (2.6)

$$\dot{l}[\eta(x)]\,\eta'(x) = 1, \quad \eta'[l(t)]\,\dot{l}(t) = 1 \qquad \ldots(2.7)$$

where $\dot{l}(t) = \dfrac{dl}{dt}, \quad \eta'(x) = \dfrac{d\eta}{dx}, \quad \eta'[l(t)] = \dfrac{d\eta}{d[l(t)]}. \qquad \ldots(2.8)$

Obviously $\dfrac{dx}{dt} = \dot{l}(t)$ is the load speed at t, and $\dfrac{dx}{dt} = \dfrac{1}{\eta'(x)}$ is the load speed at x.

The Laplace transform of a function $v(x, z, t)$ with respect to t will be defined by

$$L[v(x,z,t)] = V(x,z,s) = \int_0^\infty e^{-st}\, v(x,z,t)\, dt \qquad \ldots(2.9)$$

so that $v(x,z,t) = L^{-1}[V(x,z,s)]$. ...(2.10)

Assuming that the medium is at rest prior to the instant $t = 0$ and applying Laplace transform to eqns. (2.1) and (2.2) we get

$$\frac{\partial^2 V_1}{x\partial^2} + \frac{\partial^2 V_1}{\partial z^2} = \frac{S^2}{\beta_1^2} V_1 \qquad \ldots(211)$$

$$\frac{\partial^2 V_2}{\partial^2 x^2} + \frac{\partial^2 V_2}{\partial z^2} = \frac{S^2}{\beta_2^2} V_2. \qquad \ldots(2.12)$$

The Foruier transform of a function $v(x, z, s)$ with respect to x will be written as $\overline{V}(\xi, z, s)$, where

$$\overline{V}(\xi, z, s) = \int_{-\infty}^{\infty} V(x, z, s)\, e^{i\epsilon x}\, dx \qquad \ldots(2.13)$$

$$V(x, z, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{V}(\xi, z, s)\, e^{i\xi x}\, d\xi\,. \qquad \ldots(2.14)$$

Applying Fourier transform to eqns. (2.11) and (2.12) we can remove the variable x and reduce the partial differential equations to ordinary differential equations given by

$$\frac{d^2\overline{V}_1}{dz^2} - \left(\xi^2 + \frac{S^2}{\beta_1^2}\right)\overline{V}_1 = 0 \qquad \ldots(2.15)$$

and $$\frac{d^2\overline{V}_2}{dz^2} - \left(\xi^2 + \frac{S^2}{\beta_2^2}\right)\overline{V}_2 = 0 \qquad \ldots(2.16)$$

Since $v_2 \to 0$ as $z \to \infty$, the appropriate solutions of eqns. (2.15) and (2.16) are

$$\overline{V}_1 = A\cosh a_1 z + C\sinh a_1 z, \quad (-h \leqslant z \leqslant 0) \qquad \ldots(2.17)$$

$$\overline{V}_2 = D\, e^{-a_2 z}, \qquad (z \geqslant 0) \qquad \ldots(2.18)$$

where

$$a_1 = \sqrt{\xi^2 + \frac{S^2}{\beta_1^2}}, \quad a_2 = \sqrt{\xi^2 + \frac{S^2}{\beta_2^2}}\,. \qquad \ldots(2.19)$$

Taking Laplace and Fourier transforms of the boundary conditions (2.3), (2.4) and (2.5), we get

$$\bar{V}_1 = \bar{V}_2, \quad \text{at } z = 0 \qquad \text{...(2.20)}$$

$$\frac{d\bar{V}_1}{dz} = 0, \quad \text{at } z = -h \qquad \text{...(2.21)}$$

$$\mu_1 \frac{d\bar{V}_1}{dz} - \mu_2 \frac{d\bar{V}_2}{dz} = \bar{F}, \quad \text{at } z = 0. \qquad \text{...(2.22)}$$

Using the values of $\bar{V}_1$ and $\bar{V}_2$ from (2.17) and (2.18) in the eqns. (2.20), (2.21) and (2.22) we easily obtain $A = D$, ...(2.23)

$$A \sinh a_1, h = C \cosh a_1, h \qquad \text{...(2.24)}$$

$$\mu_1 a_1 \ C + \mu_2 a_2 \ D = \bar{F}. \qquad \text{...(2.25)}$$

Solving eqns. (2.23), (2.24) and (2.25) for A, D and C, we get

$$A = D = \frac{\bar{F} \cos\text{h}\, a_1 h}{\mu_1 a_1 \sin\text{h}\, a_1 h + \mu_2 a_2 \cosh a_1 h} \qquad \text{...(2.26)}$$

$$C = \frac{\bar{F} \sin\text{h}\, a_1 h}{\mu_1 a_1 \sin\text{h}\, a_1 h + \mu_2 a_2 \cosh a_1 h} \; . \qquad \text{...(2.27)}$$

Substituting these values of A and C in (2.17), we get the transformed version of the displacement v_1, everywhere in the layer. However we will be interested in determining the surface displacement only. Thus putting $z = -h$ in this transformed version, we obtain

$$\bar{V}_1 (\xi, -h, s) = \frac{\bar{F}}{\mu_1 a_1 \sinh a_1 h + \mu_2 a_2 \cos\text{h}\, a_1 h} \; . \qquad \text{...(2.28)}$$

But

$$\mu_1 a_1 \sinh a_1 h + \mu_2 a_2 \cosh a_1 h = \frac{(\mu_1 a_1 + \mu_2 a_2) e^{a_1 h}}{2} \left[1 - \frac{(\mu_1 a_1 - \mu_2 a_2) e^{-2a_1 h}}{\mu_1 a_1 - \mu_2 a_2} \right] \qquad \text{...(2.29)}$$

Hence

$$\bar{V}_1 (\xi, -h, s) = \frac{2\bar{F} e^{-a_1 h} (1 - k e^{-2a_1 h})^{-1}}{\mu_1 a_1 + \mu_2 a_2} \qquad \text{...(2.30)}$$

where $K = \dfrac{\mu_1 a_1 - \mu_2 a_2}{\mu_1 a_1 + \mu_2 a_2}$, $(K < 1)$, ...(2.31)

Equation (2.30) gives on making use of (2.14) ...(2.32)

$$V_1 (x, -h, s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\bar{F} e^{-a_1 h} e^{-i\xi x} (1 - K e^{-2a_1 h})^{-1}}{\mu_1 a_1 + \mu_2 a_2} \, d\xi \qquad \text{...(2.33)}$$

Making the substitution $\xi = iks$, we find

$$a_1 = sb_1 \;, \quad a_2 = sb_2, \qquad \text{...(2.34)}$$

where $b_1 = \sqrt{\dfrac{1}{\beta_1^2} - k^2}$, $b_2 = \sqrt{\dfrac{1}{\beta_2^2} - k^2}$ . ...(2.35)

Equation (2.32) then can be written as

$$V_1(x,-h,s) = \frac{-i}{\pi}\int_{-i\infty}^{i\infty} \frac{\bar{F} e^{-shb_1+ksx}(1-Ke^{-2sb_1h})^{-1}}{\mu_1 b_1+\mu_2 b_2}\, dk. \quad \ldots(2.36)$$

3. Concentrated Moving Discontinuity

Here we take

$$f(x,t) = M\,\delta[x-l(t)] \quad \ldots(3.1)$$

where M is a constant and $\delta(x)$ is the Dirac's delta function, Since $l(o) = 0$ this shows that a shearing stress discontinuity is assumed to occur suddenly at the instant $t = 0$, at $x = 0$, and then travels with nonuniform velocity in the x-direction.

Equation (3.1) gives on taking Laplace transform

$$F = \int_0^\infty M\,\delta[x-l(t)]\, e^{-st}\, dt \quad \ldots(3.2)$$

But $\delta[x-l(t)] = \eta'(x)\,\delta[t-\eta(x)]$. ...(3.3)

Hence $F = \int_9^\infty M\,\eta'(x)\,\delta[t-\eta(x)]\,e^{-st}\,dt$ ...(3.4)

$$= M\,\eta'(x)\,e^{-s\eta(x)}\,H[\eta(x)]. \quad \ldots(3.5)$$

Equation (3.5) gives on taking Fourier transform

$$\bar{F} = \int_{-\infty}^{\infty} M\,\eta'(x)\,e^{-s\eta(x)}\,H[\eta(x)]\,e^{i\xi x}\,dx\,. \quad \ldots(3.6)$$

Putting $\xi = iks$, we get

$$\bar{F} = M\int_0^\infty \eta'(p)\,e^{-s\eta(p)}\,e^{-ksp}\,dp \quad \ldots(3.7)$$

where we have changed the variable of integration from x to p and made use of the result

$$\left.\begin{array}{l} H[\eta(x)] = 1,\ \eta(x) > 0\,,\ x > 0 \\ H[\eta(x)] = 0,\ \eta(x) \leqslant 0\,,\ x \leqslant 0 \end{array}\right\}. \quad \ldots(3.8)$$

Putting the value of $\bar{F}$ from eqn. (3.7) into eqn. (2.36) we get

$$V_1(x,-h,s) = \frac{-Mi}{\pi}\int_0^\infty \eta'(p)\,e^{-s\eta(p)}\int_{-i\infty}^{i\infty} \frac{e^{-s(b_1h-kx+kp)}(1-K\,e^{-2sb_1h})^{-1}}{\mu_1 b_1+\mu_2 b_2}\,dk\,dp.$$

Expanding the term $(1-K\,e^{-2sb_1h})^{-1}$ as infinite series using binomial theorem, we get

$$V_1(x,-h,s) = \frac{-Mi}{\pi}\int_0^\infty \eta'(p)\,e^{-s\eta(p)}\int_{-i\infty}^{i\infty} \frac{e^{-s(b_1h-kx+kp)}(1+Ke^{-sb_1h}+k^2e^{-4sb_1h}+\ldots)}{\mu_1 b_1+\mu_2 b_2}\,dk\,dp. \quad \ldots(3.9)$$

Laplace inversion of each term of the infinite series in this expression will now be carried out using de Hoop's version of Cagniard's technique.

The first term in (3.9) can be written as

$$I_1 = \frac{-Mi}{\pi}\int_0^\infty \eta'(p)\, e^{-s\eta(p)} \int_{-i\infty}^{i\infty} \frac{e^{-s(b_1h-kx+kp)}}{\mu_1 b_1+\mu_2 b_2}\, dkdp. \qquad ...(3.10)$$

We now put

$$b_1h-kx + kp = \tau \qquad ...(3.11)$$

where τ is real and positive. ...(3.12)

Solving eqn. (3.11) for k, we get

$$k = \frac{-\tau(x-p) \pm ih\sqrt{\tau^2-r_1^2\beta_1^{-2}}}{(x-p)^2+h^2}$$

$$= -\frac{\tau}{r_1}\cos\theta_1 \pm i\frac{\sin\theta_1}{r_1}\sqrt{\tau^2-r_1^2\,\beta_1^2} = k_1 \pm \text{(say)}, \qquad ...(3.13)$$

where $x-p=r_1\cos\theta_1, h = r_1\sin\theta_1, \theta_1=\tan^{-1}\frac{h}{x-p}, \quad 0 \leqslant \theta_1 \leqslant \pi,$ ...(3.14)

$$r_1 = \sqrt{(x-p)^2+h^2}. \qquad ...(3.15)$$

Equation (3.13) defines one branch of a hyperbola with vertex at $k = -\cos\theta_1/\beta_1$ and asymptotes making an angle θ_1 with negative real axis. As τ varies from r_1/β_1 to ∞, k moves away from real axis, along the hyperbolic path given by (3.13), in either the upper or lower half plane (see Figure 3).

The integrand in (3.10) has branch points at $k = \pm 1/\beta_1, k = \pm 1/\beta_2$. The k plane is out along $1/\beta_2 < |\text{Re}\ k| < \infty$, $1/\beta_1 < |\text{Re}\ k| < \infty$ and $\text{Im}\ k = 0$. Now, depending on θ_1, the hyperbola (3.13) may or may not intersect any branch cut of the integrand in (3. 10).

If θ_1 lies in the range

$$\cos^{-1}\frac{\beta_1}{\beta_2} \leqslant \theta_1 \leqslant \pi-\cos^{-1}\frac{\beta_1}{\beta_2} \quad \text{(Range } APB \text{ in Fig. 2)}, \qquad ...(3.16)$$

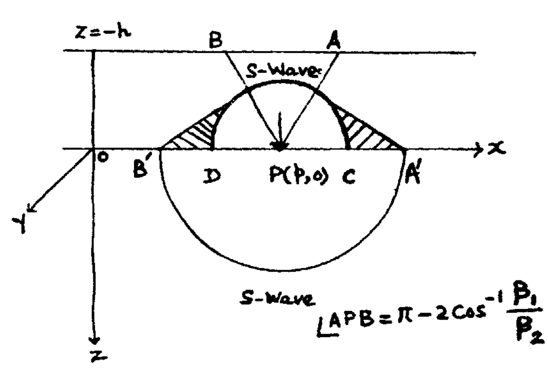

Fig. 2. Formation of head wave in the region defined by ∠ APA′ and ∠ EPB′. The region defined by ∠ APB receives only S Wave.

the hyperbola does not intersect any branch cut of the integrand as shown in Fig. 3. Since the integrand $\to 0$ as $|k| \to \infty$, applying Cauchy's theorem, equation (3.10) can be written as

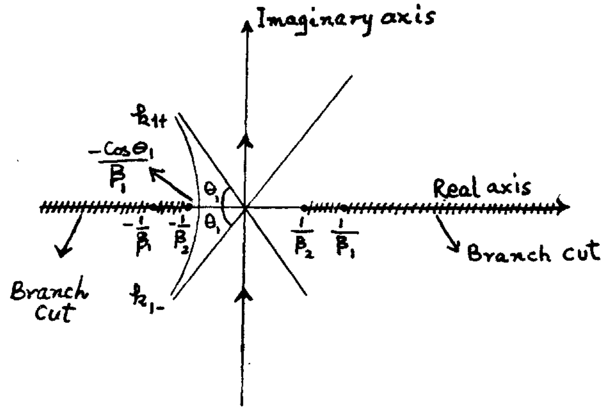

Fig. 3. Contour of integration when the hyperbola does not intersect branch cuts.

$$I_1 = -\frac{Mi}{\pi}\int_0^\infty \eta'(p)\, e^{-s\eta(p)} \int_{r_1/\beta_1}^\infty \left[\frac{e^{-s\tau}\,\frac{dk_{1+}}{d\tau}}{\mu_1 b_1(k_{1+})+\mu_2 b_2(k_{1+})} - \frac{e^{-s\tau}\,\frac{dk_{1-}}{d\tau}}{\mu_1 b_1(k_{1-})+\mu_2 b_2(k_{1-})}\right] d\tau dp \qquad \text{...(3.17)}$$

Equation (3.13) gives

$$\frac{dk_1\pm}{d\tau} = \frac{-\cos\theta_1}{r_1} \pm i\,\frac{\tau\sin\theta_1}{r_1\sqrt{\tau^2-r_1^2/\beta_1^{-2}}}. \qquad \text{...(3.18)}$$

Equations (2.35), (3.13) and (3.18) show that

$$\frac{e^{-s\tau}}{\mu_1 b_1(k_{1-})+\mu_2 b_2(k_{1-})}\;\frac{dk_{1-}}{d\tau} \text{ is the complex conjugate of}$$

$$\frac{e^{-s\tau}}{\mu_1 b_1(k_{1+})+\mu_2 b_2(k_{1+})}\;\frac{dk_{1+}}{d\tau}.$$

Hence equation (3.17) gives

$$I_1 = \frac{2M}{\pi}\int_0^\infty \eta'(p)e^{-s\eta(p)}\int_{r_1/\beta_1}^\infty \text{Im}\left[\frac{e^{-s\tau}\,dk_{1+}/d\tau}{\mu_1 b_1(k_{1+})+\mu_2 b_2(k_{1+})}\right] d\tau\, dp.$$

$$\text{Thus}\quad L^{-1}I_1 = \frac{2M}{\pi}\int_0^\infty \eta'(p)\int_{r_1/\beta_1}^\infty \text{Im}\left[\frac{dk_{1+}/d\tau}{\mu_1 b_1(k_{1+})+\mu_2 b_2(k_{1+})}\right] L^{-1}e^{-s[\tau+\eta(p)]}\, d\tau\, dp$$

$$= \frac{2M}{\pi}\int_0^\infty \eta'(p)\int_{r_1/\beta_1}^\infty \text{Im}\left[\frac{dk_{1+}/d\tau}{\mu_1 b_1(k_{1+})+\mu_2 b_2(k_{1+})}\right] \delta[t-\tau-\eta(p)]\, d\tau\, dp \qquad \text{...(3.19)}$$

$= I_{11}$ (say),

where

$$I_{11} = \begin{cases} \dfrac{2M}{\pi} \displaystyle\int_0^\infty \eta'(p)\, G_{11}(t,p)\, dp, & t \geqslant t_1^{(1)} \\ 0 & , t < t_1^{(1)} \end{cases} \qquad \ldots(3.20)$$

where $\quad G_{11}(t,p) = \operatorname{Im}\left[\dfrac{dk_{1+}/d\tau}{\mu_1 b_1(k_{1+}) + \mu_2 b_2(k_{1-})} \right], \; \tau = t - \eta(p) \qquad \ldots(3.21)$

and $\quad t_1^{(1)} = \eta(p) + \dfrac{r_1}{\beta_1}\,, \qquad \ldots(3.22)$

If θ_1 does not lie in the range indicated in (3.16), then the hyperbola crosses the portion of the branch cuts between $-1/\beta_2$ and $-1/\beta_1$ as shown in Fig. 4. The contour of integration is as shown in Fig. 4. The additional path in this case, which gives rise to head waves, consists of a circle of radius ϵ ($\epsilon \to 0$) centered at $k = -1/\beta_2$ and of two segments represented by

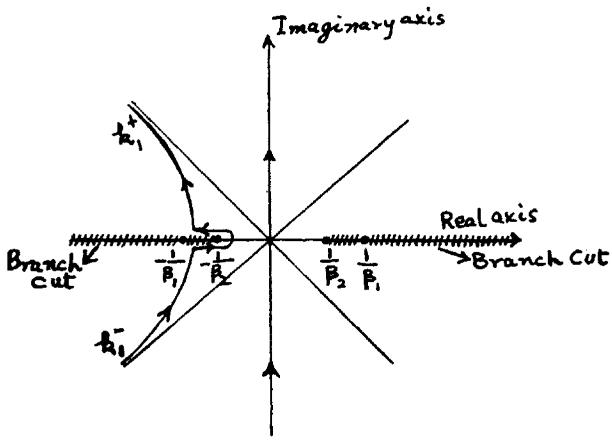

Fig. 4. Contour of integration when the hyperbola intersects branch cuts.

$$k = \left[-\frac{\tau}{r_1}\cos\theta_1 + \frac{\sin\theta_1}{r_1}\sqrt{\frac{r_1^2}{\beta_1^2} - \tau^2} \right] \pm i\epsilon = k_1 \pm (\tau) \qquad \ldots(3.23)$$

for θ_1 in the range

$$0 \leqslant \theta_1 < \cos^{-1}\frac{\beta_1}{\beta_2} \quad (x > p) \qquad \text{(see Fig. 2).} \qquad \ldots(3.24)$$

Since

$$\tau = b_1 h - k(x-p) = -k\, r_1 \cos\theta_1 + r_1 \sin\theta_1 \sqrt{\frac{1}{\beta_1^2} - k^2}$$

$$k = -\frac{\cos\theta_1}{\beta_1} \quad \text{corresponds to} \quad \tau = \frac{r_1}{\beta_1}\,,$$

$$k = -\frac{1}{\beta_2} \quad \text{corresponds to } \tau = \frac{r_1 \cos\theta_1}{\beta_2} + r_1 \sin\theta_1 \sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}}\,.$$

Hence for (3.23) the range of τ is

$$\frac{r_1}{\beta_1} \geqslant \tau \geqslant \frac{r_1 \cos\theta_1}{\beta_2} + r_1 \sin\theta_1 \sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}}\,. \quad \text{...(3.25)}$$

While the contribution of the ϵ-Circular are is nil as $\epsilon \to 0$, the path (3.23) together with the hyperbolic path (3.13) gives

$$L^{-1} I_1 = I_{11} + I_{12}, \quad \text{...(3.26)}$$

where I_{11} is defined in (3.20) and

$$I_{12} = \begin{cases} \dfrac{2M}{\pi} f_{\theta_1}^{(1)} \displaystyle\int_0^\infty \eta'(p)\, G_{12}(t,p)\, dp, & t_{12}^{(1)} \leqslant t \leqslant t_1^{(1)} \\ 0 & , \quad t < t_{12}^{(1)}\ , \quad t > t_1^{(1)} \end{cases} \quad \text{...(3.27)}$$

where $$t_{12}^{(1)} = \eta(p) + \frac{r_1 \cos\theta_1}{\beta_2} + r_1 \sin\theta_1 \sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}} \quad \text{...(3.28)}$$

$$G_{12}(t,p) = \text{Im}\left[\frac{dk_1^+/d\tau}{\mu_1 b_1(k_1^+) + \mu_2 b_2(k_1^+)}\right], \quad \tau = t - \eta(p) \quad \text{...(3.29)}$$

$$\left.\begin{aligned} f_{\theta_1}^{(1)} &= 1, \text{ for } \theta_1 \text{ in the range (3.24)} \\ f_{\theta_1}^{(1)} &= 0, \text{ otherwise.} \end{aligned}\right\} \quad \text{...(3 30)}$$

Finally for θ_1 in the range

$$\pi - \cos^{-1}\frac{\beta_1}{\beta_2} < \theta_1 \leqslant \pi, \ (x<p), \ \text{[see Fig. 2]} \quad \text{...(3.31)}$$

we have a term I_{13} similar to the term I_{12} in (3.26). The term I_{13} is given by

$$I_{13} = \begin{cases} \dfrac{2M}{\pi} f_{\theta_1}^{(2)} \displaystyle\int_0^\infty \eta'(p)\, G_{13}(t,p)\, dp, & T_{12}^{(1)} \leqslant t \leqslant t_1^{(1)}\ , \\ 0 & , \quad t < T_{12}^{(1)}\ , \quad t > t_1^{(1)}\ , \end{cases} \quad \text{...(3.32)}$$

where $$T_{12}^{(1)} = \eta(p) + \frac{r_1}{\beta_2} \,|\cos\theta_1|\, + r_1 \sin\theta_1 \sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}} \quad (= t_{12}^{(1)}) \quad \text{...(3.33)}$$

$$G_{13}(t,p) = \text{Im}\left[\frac{dk_1^+/d\tau}{\mu_1 b_1(k_1^+) + \mu_2 b_2(k_1^+)}\right], \quad \tau = t - \eta(p) \quad \text{...(3.34)}$$

$$k_1^+ = \left(-\frac{\tau}{r_1} \cos\theta_1 - \frac{\sin\theta_1}{r_1}\sqrt{\frac{r_1^2}{\beta_1^2} - \tau^2}\right) + i\epsilon \quad \text{...(3.35)}$$

$$\left.\begin{aligned} f_{\theta_1}^{(2)} &= 1, \text{ for } \theta_1, \text{ in the range (3.31)} \\ f_{\theta_1}^{(2)} &= 0, \text{ otherwise} \end{aligned}\right\} \quad \text{...(3.36)}$$

Hence $L^{-1} I_1 = I_{11} + I_{12} + I_{13}$ ...(3.37)

In general

$$L^{-1} I_n = I_{n_1} + I_{n_2} + I_{n_3}, \quad n = 1, 3, 5, \quad \text{...(3.38)}$$

where

$$I_{n1} = \begin{cases} \dfrac{2M}{\pi} \displaystyle\int_0^\infty \eta'(p)\, G_{n_1}(t,p)\, dp, & t > t_1^{(n)}, \\ 0 & , t < t_1^{(n)}, \end{cases} \qquad \ldots(3.39)$$

$$I_{n2} = \begin{cases} \dfrac{2M}{\pi} f_{\theta_n}^{(1)} \displaystyle\int_0^\infty \eta'(p)\, G_{n_2}(t,p)\, dp, & t_{12}^{(n)} \leqslant t \leqslant t_1^{(n)} \\ 0 & , t < t_{12}^{(n)},\ t > t_1^{(n)}, \end{cases} \qquad \ldots(3.40)$$

$$I_{n3} = \begin{cases} \dfrac{2M}{\pi} f_{\theta_n}^{(2)} \displaystyle\int_0^\infty \eta'(p)\, G_{n_3}(t,p)\, dp\ t, & T_{12}^{(n)} \leqslant t \leqslant t_1^{(n)} \\ 0 & , < T_{12}^{(d)},\ t > t_1^{(n)} \end{cases} \qquad \ldots(3.41)$$

$$G_{n1}(t,p) = \mathrm{Im}\left[\frac{\dfrac{dk_{n+}}{d\tau} \cdot K^{n-1/2}[k_{n+}(\tau)]}{\mu_1 \sqrt{\dfrac{1}{\beta_1^2} - k_{n+}^2} + \mu_2 \sqrt{\dfrac{1}{\beta_2^2} - k_{n+}^2}} \right], \quad \tau = t - \eta(p) \qquad \ldots(3.42)$$

$$G_{n_2}(t,p) = \mathrm{Im}\left[\frac{\dfrac{dk_{n+}}{d\tau} \cdot K^{n-1/2}\ k_n^+(\tau)}{\mu_1 \sqrt{\dfrac{1}{\beta_1} - k_n^{+2}} + \mu_2 \sqrt{\dfrac{1}{\beta_2^2} - k_n^{+2}}} \right], \quad \tau = t - \eta(p) \qquad \ldots(3.43)$$

$$G_{n_3}(t,p) = \mathrm{Im}\left[\frac{\dfrac{dk^+_n}{d\tau} \cdot K^{n-1/2}[k^+_n(\tau)]}{\mu_1 \sqrt{\dfrac{1}{\beta_1^2} - k_n^{+2}} + \mu_2 \sqrt{\dfrac{1}{\beta_2^2} - k_n^{+2}}} \right], \quad \tau = t - \eta(p) \qquad \ldots(3.44)$$

$$k_{n+} = -\frac{\tau}{r_n} \cos\theta_n + i \frac{\sin\theta_n}{r_n} \sqrt{\tau^2 - \frac{r_n^2}{\beta_1^2}} \qquad \ldots(3.45)$$

$$k_n^+ = \left[-\frac{\tau}{r_n} \cos\theta_n + \frac{\sin\theta_n}{r_n} \sqrt{\frac{r_n^2}{\beta_1^2} - \tau^2} \right] + i\epsilon \qquad \ldots(3.46)$$

$$k_n^+ = \left[-\frac{\tau}{r_n} \cos\theta_n - \frac{\sin\theta_n}{r_n} \sqrt{\frac{r_n^2}{\beta_1^2} - \tau^2} \right] + i\epsilon \qquad \ldots(3.47)$$

$$(x-p) = r_n \cos\theta_n,\ nh = r_n \sin\theta_n,\ \theta_n, = \tan^{-1} \frac{nh}{x=p} \qquad \ldots(3.48)$$

$$\left.\begin{array}{l} f_{\theta_n}^{(1)} = 1, \text{ for } \theta_n \text{ in the range } 0 \leqslant \theta_n \leqslant \cos^{-1} \dfrac{\beta_1}{\beta_2} \\ f_{\theta_n}^{(1)} = 0, \text{ otherwise} \end{array}\right\} \qquad \ldots(3.49)$$

$$\left.\begin{array}{l} f_{\theta_n}^{(2)} = 1, \text{ for } \theta_n \text{ in the range } \pi - \cos^{-1}\dfrac{\beta_1}{\beta_2} < \theta_n \leqslant \pi \\ f_{\theta_n}^{(2)} = 0, \text{ otherwise} \end{array}\right\} \qquad \ldots(3.50)$$

$$t_{1}^{(n)} = \eta\,(p) + \frac{r_n}{\beta_1} \qquad \ldots(3.51)$$

$$t_{12}^{(n)} = \eta\,(p) + \frac{r_n \cos\theta_n}{\beta_2} + r_n \sin\theta_n \sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}} \qquad \ldots(3.52)$$

$$T_{12}^{(n)} = \eta\,(p) + \frac{r_n \,|\cos\theta_n\,|}{\beta_2} + r_n \sin\theta_n \sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}}\,. \qquad \ldots(3.53)$$

Equations (3.9) and (3.38) give

$$v_1\,(x-h,\,t) = \sum_{n=1,3,5,\cdots} L^{-1}\,[I_{n_1} + I_{n_2} + I_{n_3}]\,. \qquad \ldots(3.54)$$

The terms for different values of n, $(n > 1)$,, in (3.51) are associated with S-waves and head waves undergoing repeated reflections in the upper layer. The terms containing $f_{\theta_n}^{(1)}$ and $f_{\theta_n}^{(2)}$ are due to head waves.

4. Expanding Discontinuity

Here we take

$$f\,(x,\,t) = P\,[H(x) - H\,\{x - l(t)\}], \qquad \ldots(4.1)$$

where P is a constant.

This shows that the stress discontinuity is created at $x = 0$ and then expands nonuniformly in the x-direction.

$$f_1 = P\,[\delta(x) - \delta\,\{x - l\,(t)\}] \qquad \ldots(4.2)$$

where $f_1 = \dfrac{\partial f}{\partial x}$. $\qquad \ldots(4.3)$

On taking Laplace and Fourier transforms, eqn. (4.3) gives, as in previous section,

$$\bar{F_1} = \frac{P}{s} - P \int_0^\infty \eta'\,(p)\, e^{-s\eta(p)}\, e^{i\xi p}\, dp \qquad \ldots(4.4)$$

$$= \frac{P}{s} - P \int_0^\infty \eta'\,(p)\, e^{-s\eta(p)}\, e^{-kps}\, dp \qquad \ldots(4.5)$$

where $\xi = iks$.

Differentiating the boundary condition (2.5) with respect to x, we get

$$\mu_1 \frac{\partial^2 v_1}{\partial x \partial z} - \mu_2 \frac{\partial^2 v_2}{\partial x \partial z} = f_1\; H(t), \text{ at } z = 0. \qquad \ldots(4.6)$$

Taking Laplace and Fourier transforms this gives

$$\mu_1 \frac{d\bar{V}}{dz} - \mu_2 \frac{d\bar{V}_2}{dz} = \frac{\bar{F_1}}{-i\xi}\,, \text{ at } z = 0\,. \qquad \ldots(4.7)$$

Comparing (2.22) with (4.7), we see that in this case $\bar{F}$ is to be replaced by $\bar{F_1}/-i\xi$. Hence replacing $\bar{F}$ by $\bar{F_1}/-i\xi$ in equation (2.36) and using the value of $\bar{F_1}$ from (4.5), we get

$$V_1(x,-h,s) = \frac{-i}{\pi}\int_{-i\infty}^{i\infty} \frac{\left[\frac{P}{ks^2} - \frac{P}{ks}\int_0^\infty \eta'(p)\, e^{-s\eta(p)}\, e^{-ksp}\, dp\right] e^{-shb_1+ksx}\,(1-ke^{-2sb_1h})^{-1}}{\mu_1 b_1+\mu_2 b_2}\, dk. \qquad \ldots(4.8)$$

Since integration by parts gives

$$\frac{P}{ks^2} - \frac{P}{ks}\int_0^\infty \eta'(p)\, e^{-s\eta(p)}\, e^{-ksp}\, dp = \frac{P}{s}\int_0^\infty e^{-s[\eta(p)+kp]}\, dp, \qquad \ldots(4.9)$$

equation (4.8) becomes

$$V_1(x,-h,s) = -\frac{Pi}{\pi}\int_0^\infty \frac{e^{-s\eta(p)}}{s}\int_{-i\infty}^{i\infty} \frac{e^{-shb_1+kxs-ksp}\,(1-ke^{-2sb_1h})^{-1}}{\mu_1 b_1+\mu_2 b_2}\, dk\, dp \qquad \ldots(4.10)$$

Expanding the term $(1-k\,e^{-sb_1h})^{-1}$ as an infinite series, using binomial theorem, we get the first term in (4.10) as

$$I^1 = \frac{-Pi}{\pi}\int_0^\infty \frac{e^{-s\eta(p)}}{s}\int_{-i\infty}^{i\infty} \frac{e^{-s(b_1h-kx+kp)}}{\mu_1 b_1+\mu_2 b_2}\, dk\, dp\ . \qquad \ldots(4.11)$$

If we put, as in previous section,

$b_1 h - ks + kp = \tau$, where τ is real and positive, then

$$k = -\frac{\tau}{r_1}\cos\theta_1 \pm i\,\frac{\sin\theta_1}{r_1}\sqrt{\tau^2-\frac{r_1^2}{\beta_1^2}} = k_{1\pm}, \text{ where}$$

$$x-p = r_1\cos\theta_1,\ h = r_1\sin\theta_1,\ \theta_1 = \tan^{-1}\frac{h}{x-p},\quad 0\leqslant\theta_1\leqslant\pi.$$

If θ_1 lies in the range $\cos^{-1}\frac{\beta_1}{\beta_2} \leqslant \theta_1 \leqslant \pi - \cos^{-1}\frac{\beta_1}{\beta_2}$, then taking steps analogous to those used in deriving eqn. (3.19), we get

$$L^{-1}I' = \frac{2P}{\pi}\int_0^\infty\int_{r_1/\beta_1}^\infty \left[\mathrm{I_m}\, \frac{\frac{dk_{1+}}{d\tau}}{\mu_1 b_1(k_{1+})+\mu_2 b_2(k_{1+})}\right]\left[\int_0^t H(t-q)\,\delta\{q-\tau-\eta(p)\}\,dq\right] d\tau\, dp$$

$$= I^{11} \text{ (say)}, \qquad \ldots(4.12)$$

where

$$I^{11} = \begin{cases} \dfrac{2P}{\pi}\displaystyle\int_0^\infty\int_0^t H(t-q)\,G_{11}(q,p)\,dq\,dp, & q > t_1^{(1)} \\ 0 & ,\ q < t_1^{(1)}. \end{cases} \qquad \ldots(4.13)$$

G_{11} and $t_1^{(1)}$ are same as defined in previous section. If θ_1 lies in the range

$0 \leqslant \theta_1 < \cos^{-1} \frac{\beta_1}{\beta_2}$, then taking steps analogous to those used in deriving eqn. (3.26), we get

$$L^{-1} \overset{1}{I} = \overset{11}{I} + \overset{12}{I} \quad \ldots(4.14)$$

where $\overset{11}{I}$ is defined in (4.13), and

$$\overset{12}{I} = \begin{cases} \frac{2P}{\pi} f_{\theta_1}^{(1)} \int_0^\infty \int_0^t H(t-q)\, G_{12}(q,p)\, dq\, dp, & t_{12}^{(1)} \leqslant q \leqslant t_1^{(1)} \\ 0 & , q < t_{12}^{(1)},\ q > t_1^{(1)} \end{cases} \quad \ldots(4.15)$$

G_{12}, $t_{12}^{(1)}$, k_1^{+} and $f_{\theta_1}^{0(1)}$ are same as defined in previous section. Finally if θ_1 lies in the range $\pi - \cos^{-1} \frac{\beta_1}{\beta_2} < \theta_1 \leqslant \pi$, then we have a term $\overset{13}{I}$ similar to the term $\overset{12}{I}$ in (4.14). The term $\overset{13}{I}$ is given by

$$\overset{13}{I} = \begin{cases} \frac{2P}{\pi} f_{\theta_1}^{(2)} \int_0^\infty \int_0^t H(t-q)\, G_{13}(q,p)\, dq\, dp, & T_{12}^{(1)} \leqslant q \leqslant t_1^{(1)}, \\ 0 & , q < T_{12}^{(1)},\ q > t_1^{(1)}, \end{cases} \quad \ldots(4.16)$$

where G_{13}, $T_{12}^{(1)}$, k_1^{+} and $f_{\theta_1}^{(2)}$ are same as defined in previous section. Hence $T_{12}\ t_1^{(1)}$

$$L^{-1} \overset{1}{I} = \overset{11}{I} + \overset{12}{I} + \overset{13}{I} \quad \ldots(4.17)$$

In general

$$L^{-1} \overset{n}{I} = \overset{n_1}{I} + \overset{n_2}{I} + \overset{n_3}{I},\ n = 1,3,5,\ldots \quad \ldots(4.18)$$

where

$$\overset{n}{I}_1 = \begin{cases} \frac{2P}{\pi} \int_0^\infty \int_0^t H(t-q) G_{n_1}(q,p)\, dq\, dp, & q \geqslant t_1^{n} \\ 0 & , q < t_1^{(n)} \end{cases} \quad \ldots(4.19)$$

$$\overset{n}{I}_2 = \begin{cases} \frac{2P}{\pi} f_{\theta_n}^{(1)} \int_0^\infty \int_0^t H(t-q) G_{n_2}(q,p)\, dq\, dp, & t_{12}^{(n)} \leqslant q \leqslant t_1^{(n)} \\ 0 & , q < t_{12}^{(n)},\ q > t_1^{(n)} \end{cases} \quad \ldots(4.20)$$

$$\overset{n}{I}_3 = \begin{cases} \frac{2P}{\pi} f_{\theta_n}^{(2)} \int_0^\infty \int_0^t H(t-q)\, G_{n_3}(q,p)\, dq\, dp, & T_{12}^{(n)} \leqslant q \leqslant t_1^{(n)} \\ 0 & , q < T_2^{(n)} q > t_1^{(n)} \end{cases} \quad \ldots(4.21)$$

G_{n_1}, G_{n_2}, G_{n_3}, θ_n, $f^{(1)}_{\theta_n}$, $f^{(2)}_{\theta_n}$, $t_1^{(n)}$, $t_{12}^{(n)}$, and $T_{12}^{(n)}$ are same as defined in previous section

Equations (4.10) and (4.18) give

$$v_1(x,-h,t) = \sum_{n=1,3,5\ldots} L^{-1}\overset{n}{I}$$

$$= \sum_{n=1,3,5\ldots} L^{-1}(\overset{n_1}{I} + \overset{n_2}{I} + \overset{n_3}{I}) \qquad \ldots (4.22)$$

The interpretation of $f^{(1)}_{\theta_n}$ and $f^{(2)}_{\theta_n}$ is same as in previous section.

5. Derivation of Results for Uniform Motion

If the two types of discontinuities discussed in previous sections move uniformly with velocity U, then

$$\eta(p) = \frac{p}{v}$$

so that

$$\eta'(p) = \frac{1}{v}.$$

Putting these values of $\eta(p)$ and $\eta'(p)$ in (3.51) and (4.22), we get the corresponding results for uniformly moving discontinuities. The results, thus obtained, are true for all values of U.

Acknowledgement

The authors are thankful to Prof. S.J. Singh, for his keen interest and valuable suggestions. One of them (J.P.M) is highly grateful to the authorities of Vaish College, Rohtak, for their encouragement.

References

Freund, L.B. (1973). The response of an elastic solid to nonuniformly moving surface load. *J. Appl. Mech.*, 40, 699–704.

Lamb, H. (1904). On the propagation of tremors over the surface of an elastic solid. *Phil. Trans. R. Soc.*, A **203**, 1–42.

Nag, K.R. (1963). Generation of SH-type of waves due to a stress discontinuity in a layered medium. *Qt. Jl. Mech. appl. Math.*, **16**, 295–303.

Roy, A. (1979). Response of an elastic solid to nonuniformly expanding surface loads. *Int. J. Engng. Sci.*, **17**, 1023–38.

Sudo, Ken. (1972). Radiation of seismic waves from a propagating source near the free surface. *J. Phys. Earth*, **20**, 127–45.

Watanabe, K. (1977). Transient response of an elastic half space subjected to a reciprocating anti-plane shear load. *Int. J. Solids Struct.*, **13**, 63–74.