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ON THE ABSOLUTE EULER SUMMABILITY OF A FACTORED FOURIER SERIES

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In this paper a theorem on absolute Euler summability factor has been established. This theorem is an improvement of an earlier result due to Tripathy (1973). The second theorem establishes the fact that the summability factor taken in the first theorem is best possible in a certain sense.

§ 1. *Definition* 1—Let Σu_n be an infinite series with the sequence of partial sums $\{U_n\}$. The Euler's series-to-series transform

$$b_n = (q+1)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} u_\nu, \quad q>0.$$

If Σb_n is convergent, we say that Σu_n (or the sequence $\{U_n\}$) is summable (E,q), $q>0$, in short we write

$$\Sigma u_n \in (E,q) \text{ [or } \{U_n\} \in | E, q |], \quad q>0.$$

If Σb_n is absolutely convergent, we say that Σu_n (or the sequence $\{U_n\}$ is summable (E,q), $q>0$, in short we write

$$\Sigma u_n \in | E,q | \text{ [or } \{U_n\} \in | E,q |], \quad q>0.$$

Definition 2—The series Σu_n is said to be summable $| E_\alpha |$ $(0<\alpha<1)$, if

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} U_n$$

and

$$\Sigma | t_n - t_{n-1} | < \infty. \qquad \ldots(1.1)$$

Since

$$\tau_n = \sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} \nu u_\nu = n(t_n - t_{n-1}),$$

(1.1) is equivalent to

$$\sum \frac{| \tau_n |}{n} < \infty. \qquad \ldots(1.2)$$

It is easy to notice (Kwee 1972) that Definition 1 reduces to Definition 2, with the substitution $q = \frac{1-\alpha}{\alpha}$, $0<\alpha<1$, i e.

$$\left| E, \frac{1-\alpha}{\alpha} \right| \sim \left| E_\alpha \right|, \ 0 < \alpha < 1.$$

It is also known (Kwee 1972 and Tripathy 1973) that the E_α means given above can be derived from Hausdorff means.

Let $f(t)$ be Lebesgue integrable over $(-\pi,\pi)$ and periodic with period 2π and let

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \qquad \ldots(1.3)$$

We write through out:

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \ \ \alpha>0,$$

$$\Phi_0(t) = \phi(t)$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t), \ \ (\alpha \geqslant 0)$$

$$\Delta \mu_n = \mu_n - \mu_{n+1}.$$

Denoting the greatest integer contained in x by $[x]$, we write

$$M = [t^{-1}], \ N = [t^{-2}].$$

By "$F(t) \in BV(h,K)$" we mean that $F(t)$ is of bounded variation over (h, k). By "$\Sigma u_n \in P$" we mean that Σu_n is summable by method P. Analoguously if Σu_n is absolutely summable by method P we write $\Sigma u_n \in |P|$. We write 'A' for a positive constant which is not necessarily the same at each occurrence.

The absolute Euler summability of a Fourier series have been studied by Mohanty and Mohapatra (1968) and Kwee (1972) independently and their results reads as follows :

Theorem A—If $\phi(t) \log t^{-1} \in BV(0,\delta)$, $0 < \delta < 1$, then

$$\sum_{n=0}^{\infty} A_n(x) \in |E,q|, \ q > 0.$$

we state below a result on absolute Euler summability of a Factored Fourier series due to Tripathy (1973).

Theorem B—If $\phi_\alpha(t) \in BV(0,\pi)$, $0<\alpha<\frac{3}{2}$, then $\sum_{n=1}^{\infty} n^{-\lambda} A_n(x) \in |E, q|$, $q>0$,

where $\lambda > \max(\alpha-\frac{1}{2}, \frac{1}{2})$.

Remarkz on Theorem B: (I) It is evident that the condition imposed on the generating function in Theorem B is non-local for $0<\alpha<1$ and local for $1 \leqq \alpha < \frac{3}{2}$.

From Theorem A, it is apparent that $|E, q|$, $q>0$ summability of $\sum_{n=0}^{\infty} A_n(x)$ is a local property of the generating function and hence Theorem B should hold under local conditions.

(II) Since $\phi_\alpha(t) \in BV(0,\pi) \Rightarrow \phi_\beta(t) \in BV(0,\pi)$ for $\beta>\alpha$ it is expected that the absolute summability factor $1/n^\lambda$ taken in Theorem B should be altered accordingly when ever there is a change in the value of α. This is not so in case of Theorem B as it is seen that the factor remains fixed (the value being $n^{-(\frac{1}{2})-\epsilon}$, $\epsilon>0$) for $0<\alpha\leq 1$.

(III) There is no point in restricting α in the interval $(0,\frac{3}{2})$ and as such the restriction seems to be artificial.

The object of the present work is to remove the defects cited above in (I), (II) and (III) by proving the following theorem.

Theorem 1—If $\phi_\alpha(t) \in BV(0,\delta)$, $\delta>0$, then $\Sigma A_n(x)/n^{\alpha/2} \in |E,q|$, $q>0$ for $0<\alpha\leq 2$.

Remarks on the Theorem 1: There is no point in proving the theorem for $\alpha>2$ as the series $\Sigma A_n(x)/n^{\alpha/2}$ is absolutely convergent for $\alpha>2$ when ever $\phi(t) \in L(0,\pi)$. The choice of the factor $n^{-\alpha/2}$ in Theorem 1 is best possible for $\alpha=1$, i.e. the factor $n^{-1/2}$ can not be replaced by $n^{-(\frac{1}{2})+\epsilon}$ $(\epsilon>0)$ in theorem 1 when $\alpha=1$. This is justified by proving

Theorem 2—There exists a function $f(t)$ of class L such that $\phi_1(t) \in BV(0,\pi)$ but the series $\Sigma A_n(x)/n^{\frac{1}{2}-\epsilon}$, $\epsilon>0$ is not summable $|E,q|$, $q>0$.

§ 2. *Proof of Theorem* 1—(i) NOTATION:

$$u_\nu(n)=\binom{n}{\nu} q^{n-\nu}(1+q)^{-n}$$

$$\begin{matrix} g_s(n,t,\beta) \\ g_c(n,t,\beta) \end{matrix} = \sum_{\nu=1}^{n} u_\nu(n)\nu^\beta \begin{matrix} \sin t \\ \cos t \end{matrix} \quad \text{for all real } \beta.$$

$$J(n,u) = \frac{1}{\Gamma(1+h-\alpha)}\int_u^\delta (t-u)^{h-\alpha}(d/dt)^{h+1}\, g_s(n,t,-1-\alpha/2)\,dt$$

where $h=[\alpha]$.

$$\rho^2(t) = \frac{1+q^2+2q\cos t}{(1+q)^2} = 1-(2\sqrt{q}\sin\tfrac{1}{2}t/(1+q))^2$$

$$\tan\theta = \frac{q\sin t}{q+\cos t}.$$

We need the following estimates for the proof of Theorem 1.

$$g_s(n,t,\beta) = O\{(\rho(t))^n\} \text{ for } \beta=0. \quad \ldots(2.1)$$

$$g_s(n,t,\beta) = \begin{cases} O\{(n^\beta)\} \\ O(t^{-1}n^{\beta-1}) + O(n^\beta\rho^n(t)),\ \beta\neq 0 \end{cases} \quad \ldots(2.2)$$

$$(d/dt)^r g_s(n,t,\beta) = \begin{cases} O(n^{\beta+r}) \\ O(t^{-1}n^{\beta+r-1}) + O(n^{\beta+r}\rho^n(t)). \end{cases} \quad \ldots(2.3)$$

The estimations (2.1), (2.2) and (2.3) remains valid if $g_s(n,t,\beta)$ is replaced by $g_c(n,t,\beta)$. For $0<\alpha<2$.

$$J(n,u)=\begin{cases} O\,(n^{(\alpha-2)/2}) \\ O\,(n^{(\alpha-2)/2}\rho^n(u)) + O(n^{(\alpha-4)/2}\,u^{-1}). \end{cases} \quad ...(2.4)$$

For $0<\alpha<2$ and $0<\delta<\pi$,

$$J(n,\delta)=\begin{cases} O\,(n^{(\alpha-2)/2}) \\ O\,(n^{(\alpha-2)/2}\rho^n(\delta)) + O(n^{(\alpha-2)/2}). \end{cases} \quad ...(2.5)$$

PROOF OF (2.1):—We have

$$g_c(n,t,0)=\sum_{\nu=1}^{n} u_\nu(n)\cos\nu t=(1+q)^{-n}\,Rl\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}e^{i\nu t}-\left(\frac{q}{1+q}\right)^n$$

$$=\rho^n(t)\cos n\theta-(q/(1+q))^n=O\{\rho^n(t)\},$$

as $\rho(t)$ can be made greater than $(q/(1+q))^n$ by taking sufficiently small t.

PROOF OF (2.2): We shall consider the case $\beta>0$ and $\beta<0$ separately. When $\beta>0$

$$g_s(n,t,\beta)=\sum_{\nu=1}^{n} u_\nu(n)\nu^\beta\sin\nu t=n^\beta \max_{1\leqslant N,\,N'\leqslant n}\sum_{N}^{N'}u_\nu(x)$$

$$<n^\beta\sum_{\nu=1}^{n}u_\nu(n)<n^\beta.$$

The case $\beta<0$.

Suppose that $\beta=-\gamma$, where $\gamma>0$ and $\mu=[\gamma]+1$.

Since

$$u_\nu(n)n^\beta=\frac{\left(\binom{n}{\nu}q^{n-\nu}\nu^\beta\right)}{(1+q)^n}$$

$$=\frac{(1+q)^\mu}{(n+1)(n+2)\dots(n+\mu)}\left[\frac{\binom{n+\mu}{\nu+\mu}q^{n+\mu-(\nu+\mu)}}{(1+q)^{n+\mu}}\right]\frac{(\nu+1)(\nu+2)\dots(\nu+\mu)}{\nu^\gamma}$$

$$=\frac{(1+q)^\mu}{(n+1)(n+2)\dots(n+\mu)}\,u_{\nu+\mu}(n+\mu)\,\frac{(\nu+1)(\nu+2)\dots(\nu+\mu)}{\nu^\gamma},$$

we obtain (as $\mu>\gamma$)

$$g_s(n,t,\beta)\leqq\frac{(1+q)^\mu}{(n+1)(n+2)\dots(n+\mu)}\,\frac{(n+1)(n+2)\dots(n+\mu)}{n^\gamma}\sum_{\nu=1}^{n}u_{\nu+\mu}(n+\mu)$$

$$=\frac{(1+q)^\mu}{n^\gamma}\sum_{\nu=0}^{n+\mu}u_\nu(n+\mu)=\frac{(1+q)}{n^\gamma}\sum_{\nu=0}^{n+\mu}u_\nu(n+\mu)$$

$$=\frac{(1+q)^\mu}{n^\gamma}=O\,(n^\beta).$$

This completes the proof of first part of (2.2). By partial summation, we get

$$g_s(n,t,\beta) = \sum_{\nu=1}^{n-1} \Delta(\nu^\beta) \sum_{k=1}^{\nu} u_k(n) \sin kt + n^\beta \sum_{\nu=1}^{n} u_\nu(n) \sin \nu t \qquad \ldots(2.6)$$

$$= \sum_{\nu=1}^{n-1} \Delta(\nu^\beta) \sum_{k=1}^{\nu} u_k(n) \sin kt + n^\beta \; g_s(n,t,0)$$

$$= \Sigma + n^\beta \; g_s(n,t,0), \text{ say.}$$

The largest $u\,(n_\nu)$ is $u_M(n)$ where $M = \left[\frac{n+1}{q+1}\right]$ (see Hardy (1949), p. 201). Hence $u_\nu(n)$ is increasing for $1 \leqq \nu \leqq M$ and decreasing for $M < \nu \leqq n$. Now we write

$$\sum = \sum_{\nu=1}^{M} + \sum_{\nu=M+1}^{n} = \sum_1 + \sum_2 \text{ , say.}$$

$$\sum_1 \leqq \sum_{y=1}^{M} \left| \Delta(\nu^\beta) \right| u_\nu(n) \max_{1 \leqq L, L' \leqq \nu} \left| \sum_{L}^{L'} \sin kt \right|$$

$$\leqq A\, t^{-1} \sum_{\nu=1}^{M} \nu^{\beta-1}\, u_\nu(n) \leqq A t^{-1}\, n^{\beta-1}, \qquad \ldots(2.7)$$

since the last sum can be shown to be of order $O(n^{\beta-1})$ by adopting the line of argument employed in proving the estimate of (2.2).

$$\sum_2 = \sum_{\nu=M+1}^{n} \Delta(\nu^\beta) \sum_{k=1}^{\nu} u_k(n) \sin kt$$

$$= \sum_{\nu=M+1}^{n} g_s(n,t,0) - \sum_{\nu=M+1}^{n} \Delta\,(\nu)^\beta \sum_{k=\nu+1}^{n} u_k(n) \sin kt$$

$$= O\{n^\beta \; g_s(n,t,0)\} - \Sigma_{2,2}, \text{ say.} \qquad \ldots(2.8)$$

Since $u_{\nu+1}$ is decreasing when $M < \nu \leqq n$, we get

$$\left| \sum_{2,2} \right| \leqq A \sum_{\nu=M+1}^{n} \left| \Delta\,(\nu)^\beta \right| u_{\nu+1}\,(n)\; t^{-1}$$

$$\leqq A t^{-1} \sum_{\nu=M+1}^{n} \nu^{\beta-1}\, u_{\nu+1}(n) \leqq A n^{\beta-1}\, t^{-1}. \qquad \ldots(2.9)$$

Combining the results (2.6)—(2.9), together we obtain

$$g_s(n,t,\beta) = O\{n^{\beta-1}\, t^{-1}\} + O\{n^\beta g_s(n,t,0)\}. \qquad \ldots(2.10)$$

The second inequality of (2.2) follows atonce by an appeal to (2.1).

PROOF OF (2.3): By formal computation, we get

$$(d/dt)^r g_s(n,t,\beta) = \begin{cases} (-1)^{1/2(r-1)} g_c(n,t,\beta+r), & \text{when } r \text{ is odd} \\ (-1)^{1/2r} g_s(n,t,\beta+r), & \text{when } r \text{ is even,} \end{cases}$$

and

$$(d/dt)^r g_c(n,t,\beta) = \begin{cases} (-1)^{1/2(r+1)} g_s(n,t,\beta+r), & \text{when } r \text{ is odd} \\ (-1)^{1/2r} g_c(n,t,\beta+r), & \text{when } r \text{ is even.} \end{cases}$$

In view of the above relations, the inequality (2.3) follows immediately by an appeal to (2.2).

PROOF OF (2.4): We have

$$(1+h-\alpha)\, J(n,u) = \int_u^\delta (t-u)^{h-\alpha} (d/dt)^{h+1} g_s(n,t,-1-\tfrac{1}{2}\alpha)dt$$

$$= \int_u^{u+n^{-1}} + \int_{u+n^{-1}}^{\delta} = I_1 + I_2, \text{ say.}$$

Using first estimation of (2.3), we get

$$I_1 = O(n^{-\alpha/2+h}) \int_u^{u+n^{-1}} (t-u)^{h-\alpha}\, dt = O(n^{(\alpha-2)/2})$$

and for $u+n^{-1} < \delta' < \delta$

$$I_2 = \int_{u+n^{-1}}^{\delta} (t-u)^{h-\alpha} (d/dt)^{h+1} g_s(n,t,-1-\tfrac{1}{2}\alpha)dt$$

$$= n^{\alpha-h} \int_{u+n^{-1}}^{\delta'} (d/dt)^{h+1} g_s(n,t,-1-\tfrac{1}{2}\alpha)dt = n^{\alpha-h} \Big[(d/dt)^h g_s(n,t,-1-\tfrac{1}{2}\alpha)\Big]_{u+n^{-1}}^{\delta'}$$

$$= O(n^{\alpha-h}.\, n^{h-(\alpha-2)/2}) = O(n^{(\alpha-2)/2}).$$

From I_1 and I_2 the first of (2.4) follows at once. Using the second estimate of (2.3), we have

$$I_1 < A\Big[\int_u^{u+n^{-1}} n^{h-\alpha/2-1} (t-u)^{h-\alpha}\, t^{-1} dt + n^{h-\alpha/2} \int_u^{u+n^{-1}} (t-u)^{h-\alpha} \rho^n(t) dt \Big]$$

$$= A\Big[n^{h-\alpha/2} u^{-1} \int_u^{\delta_1} (t-u)^{h-\alpha}\, dt + n^{h-\alpha/2} \rho^n(u) \int_u^{\delta_2} (t-u)^{h-\alpha}\, dt \Big], \; u < \delta_1,\; \delta_2 < u+n^{-1}$$

since $\rho^n(t)$ and t^{-1} are monotonic decreasing functions. Thus we get

$$I_1 = O[n^{(\alpha-2)/2} \rho^n(u)] + O[u^{-1} n^{(\alpha-2)/2}].$$

We have, for $u+n^{-1} < \delta_3 < \delta$,

$$I_2 = \int_{u+n^{-1}}^{\delta} (t-u)^{h-\alpha} (d/dt)^{h+1} g_s(n,t,-1-\tfrac{1}{2}\alpha)\, dt = n^{\alpha-h} \Big[(d/dt)^h g_s(n,t,-1-\tfrac{1}{2}\alpha)\Big]_{u+n^{-1}}^{\delta_3}$$

$$= O(n^{(\alpha-4)/2} u^{-1}) + O[n^{(\alpha-2)/2} \rho^n(u)],$$

by second of estimate (2.3).

This terminates the proof of (2.4). The estimate (2.5) is obtained by merely δ in place of u in (2.4).

(ii) We prove the theorem first for $0<\alpha<2$ and the $\alpha=2$ separately. Denoting the Euler transform of $\Sigma\, A_n(x)/n^{\alpha/2}$, $(0<\alpha\leq 2)$ by τ_n we have

$$\frac{\pi}{2}\,\tau_n=\int_0^\pi \phi(t)\,(d/dt)\,g_s(n,t,-1-\tfrac{1}{2}\alpha)\,dt$$

$$=\int_0^\delta+\int_\delta^\pi=\alpha_n+\beta_n,\ \text{say.} \qquad \ldots(2.11)$$

Integrating by parts, we have

$$\alpha_n=\left[\sum_{\rho=1}^{h}(-1)^{\rho+1}\Phi_\rho(t)\,(d/dt)^\rho\,g_s\,(n,t,-1-\tfrac{1}{2}\alpha)\right]_0^\delta+(-1)^h\int_0^\delta\Phi_h(t)(d/dt)^{h+1}$$

$$\cdot g_s(n,t,-1-\tfrac{1}{2}\alpha)\,dt, \qquad \ldots(2.12)$$

where $h=[\alpha]$.

Also

$$\int_0^\delta \Phi_h(t)\,(d/dt)^{h+1}\,g_s(n,t,-1-\tfrac{1}{2}\alpha)\,dt$$

$$=\frac{1}{\Gamma(1+h-\alpha)}\int_0^\delta (d/dt)^{h+1}\,g_s(n,t,-1-\tfrac{1}{2}\alpha)\int_0^t (t-u)^{h-\alpha}\,d\Phi_\alpha(u)$$

$$=\frac{1}{\Gamma(1+h-\alpha)}\int_0^\delta d\Phi_\alpha(u)\int_u^\delta (t-u)^{h-\alpha}\,g_s(n,t,-1-\tfrac{1}{2}\alpha)\,dt$$

$$=\int_0^\delta J(n,u)\,d\Phi_\alpha(u)=\Big[\Phi_\alpha(u)\,J(n,u)\Big]_0^\delta-\int_0^\delta\Phi_\alpha(u)\,\frac{d}{du}\,J(n,u)\,du.$$

Further since $\phi\,(+0)$ is finite,

$$\int_0^\delta \Phi_\alpha(u)\,(d/du)\,J(n,u)\,du=\frac{1}{\Gamma(\alpha+1)}\int_0^\delta u^\alpha\phi_\alpha(u)\,(d/du)\,J(n,u)\,du$$

$$=\frac{1}{\Gamma(\alpha+1)}\Big[\phi_\alpha\,(u)\int_0^u v^\alpha(d/dv)\,J(n,v)\,dv\Big]_0^\delta-\frac{1}{\Gamma(\alpha+1)}\int_0^\delta\Big\{\int_0^u v^\alpha(d/dv)\,J(n,v)\,dv\Big\}d\Phi_\alpha(u)$$

$$= \phi_\alpha(\delta)\ V(n,\delta)\ - \int_0^\delta V(n,u)\ d\phi_\alpha\ (u),$$

where $$V(n,u) = \frac{1}{\Gamma(\alpha+1)} \int_0^u v^\alpha\ (d/dv)\ J(n,v)\ dv.$$

Collectin g the results together, we have

$$\alpha_n = \Big[\sum_{\rho=1}^{h} (-1)^{\rho+1}\ \Phi_\rho(t)(d/dt)^\rho\ g_s(n,t,-1-\tfrac{1}{2}\alpha) \Big]_0^\delta + (-1)^h \Big[\Phi_\alpha(u)\ J(n,u)\ \Big]_0^\delta$$

$$+\ (-1)^{h+1}\ \phi_\alpha(\delta) V(n,\delta) + (-1)^h \int_0^\delta V(n,u)\ d\phi_\alpha(u).\ (2.3)$$

$$= O\ (n^{(-2-\alpha)/2}) + O\Big[\rho^n(\delta) n^{-\alpha/2}\Big] + (-1)^h \Big[\Phi_\alpha(u)\ J(n,u)\ \Big]_0^\delta + (-1)^{h+1}\ V(n,\delta)\phi_\alpha(\delta)$$

$$+ (-1)^h \int_0^\delta V(n,u)\ d\phi_\alpha\ (u), \qquad \ldots(2.13)$$

since the integrated part is zero for $0<\alpha<1$, and for $1 \leqq \alpha < 2$, the same is $O\ (n^{(-2-\alpha)/2}) + O\,(n^{-\alpha/2}\ \rho^n(\delta))$. In particular, we may suppose $\phi(t)=1$ for all t, in which case $(\phi)_\alpha(t)=1$ for all t, and hence

$$\alpha_n = \int_0^\delta \Big(\sum_{\nu=1}^{n} u_\nu(n)\ \frac{\cos \nu t}{\nu^{\alpha/2}} \Big) dt = g_s(n,\delta,-1-\frac{1}{2}\alpha)$$

$$= O(n^{(-2-\alpha)/2}) + O(n^{-\alpha/2}\rho^n(\delta)), \text{ by (2.2).} \qquad \ldots(2.14)$$

Hence from (2.13), we obtain (since $d\phi_\alpha(t)=0$)

$$O(n^{(-2-\alpha)/2}) + O(n^{-\alpha/2}\ \rho^n(\delta)) = O(n^{(-2-\alpha)/2}) + O(n^{-\alpha/2}\rho^n(\delta))$$

$$+ (-1)^h \frac{\delta^\alpha}{\Gamma(\alpha+1)}\ J(n,\delta) + (-1)^{h+1}\ V(n,\delta).$$

Thus we have

$$V(n,\delta) = O(n^{-1-\alpha/2}) + O(n^{-\alpha/2}\rho^n(\delta)) + \frac{\delta^\alpha}{\Gamma(\alpha+1)}\ J(n,\delta). \qquad \ldots(2.15)$$

We also have

$$\Gamma(\alpha+1)\ V(n,u) = \int_0^u v^\alpha (d/dv) J(n,v)\ dv = [v^\alpha J(n,v)]_0^u - \alpha \int_0^u v^{\alpha-1}\ J(n,v) dv$$

$$= u^\alpha\ J(n,u) - \alpha \int_0^u v^{\alpha-1}\ J(n,v) dv = O\ (u^\alpha n^{(\alpha-2)/2}) \qquad \ldots(2.16)$$

using first estimation of (2.4).

Again

$$\Gamma(\alpha+1)[V(n,\delta)-V(n,u)] = \int_u^\delta v^\alpha (d/dv)\, J(n,v)\, dv = [v^\alpha J(n,v)]_u^\delta - \alpha \int_u^\delta v^{\alpha-1} J(n,v)\; dv$$

$$= \delta^\alpha J(n,\delta) - u^\alpha J(n,u) + O\,(n^{\alpha/2-1}) \int_u^\delta (v^{\alpha-1})\rho^n(v)dv + O(n^{\alpha/2-2}) \int_u^\delta v^{\alpha-2}\, dv,$$

using second estimation of (2.4).

It is easy to see that the last integral is $O(u^{\alpha-2})$.

$$\int_u^\delta v^{\alpha-1}\rho^n(v)dv = \frac{-4}{n+2}\int_u^\delta \frac{v^{\alpha-1}}{\sin v}\frac{(1+q)^2}{4q}\frac{d}{dv}\left\{\rho^{n+2}(v)\right\}dv$$

$$\leqq \frac{(1+q)^2}{(n+2)q}\frac{\pi}{2}\int_u^\delta v^{\alpha-2}(d/dv)\{\rho^{n+2}(v)\}dv = \frac{\pi(1+q)^2}{2q(n+2)}u^{\alpha-2}\int_u^\delta (d/dv)\{\rho^{n+2}(v)\}\; dv$$

$$= O(n^{-1}u^{\alpha-2}\rho^n(u)).$$

Collecting the above results, we have

$$V(n,u) = V(n,\delta) - \frac{\delta^\alpha}{\Gamma(1+\alpha)}J(n,\delta) + \frac{u^\alpha}{\Gamma(\alpha+1)}J(n,u) + O(n^{\alpha/2-2}u^{\alpha/2-2}\rho^n(u))$$
$$+\; O(n^{\alpha/2-2}u^{\alpha-2}).$$

Using the known estimations for $V(n, \delta)$ and $J(n, \delta)$, we have

$$V(n,u) = O(n^{(\alpha-2)/2}\rho^n(\delta) + O\{(u^\alpha n^{(\alpha-2)/2}\rho^n(u)\} + O(u^{(\alpha-4)/2}n^{(\alpha-4)/2})$$
$$+\; O(n^{(\alpha-4)/2}\, u^{\alpha-2}\, \rho^n(u)). \qquad \ldots(2.17)$$

Since $d\Phi_\alpha(+0)=0$ and $\phi_\alpha(\delta)$ is finite, we obtain collecting the results of (2.13), (2.5) and (2.15).

$$\alpha_n = O(n^{(2-\alpha)/2}) + O(n^{-\alpha/2}\rho^n(\delta)) + O(n^{(\alpha-2)/2}\;\rho^n(\delta) + O(n^{(\alpha-4)/2} + (-1)^h \int_0^\delta V(n,u)\; d\phi_\alpha(u).$$

Since $\int_0^\delta |\; d\phi_\alpha\; (u)\; |$ is finite, it remains to establish the following relation

$$\sum_1 = \sum_{n=1}^\infty \frac{1}{n^{(2+\alpha)/2}} = O(1)$$

$$\sum_2 = \sum_{n=1}^\infty \frac{\rho^n(\delta)}{n^{\alpha/2}} = O(1)$$

$$\sum_3 = \sum_{n=1}^\infty \frac{\rho^n(\delta)}{n^{(2-\alpha)/2}} = O\;(1)$$

(*equation continued on p.* 705)

$$\sum_4 = \sum_{n=1}^{\infty} \frac{1}{n^{(4-\alpha)/2}} = O(1) \qquad \text{...(2.18)}$$

$$\sum_5 = \sum_{n=1}^{\infty} | V(n,u) | = O(1), \text{ for } 0<u<\delta.$$

First and fourth of (2.18) are trivially true for $0<\alpha<2$.

$$\sum_2 = \frac{\Gamma(1-\frac{1}{2}\alpha)}{[1-\rho(\delta)]^{(2-\alpha)/2}} = \frac{\Gamma(1-\frac{1}{2}\alpha)1+\rho(\delta)^{(2-\alpha)/2}}{[1-\rho^2(\delta)]^{(2-\alpha)/2}}$$

$$\leq \frac{A^{\Gamma}\left(1-\frac{1}{2}\alpha\right) 2^{(2-\alpha)/2}}{\left[\frac{4q}{(1+q)^2}\sin^2\frac{1}{2}\delta\right]^{(2-\alpha)/2}} \leq A.$$

Again

$$\sum_3 = \frac{\Gamma(\frac{1}{2}\alpha)}{[1-\rho(\delta)]^{\alpha/2}} = \Gamma(\alpha/2)\frac{[1+\rho(\delta)]^{\alpha/2}}{[1-\rho^2(\delta)]^{\alpha/2}}$$

$$\leq \frac{\Gamma\left(\frac{1}{2}\alpha\right)2^{\alpha/2}}{\frac{4q}{(1+q)^2}\sin^2\frac{1}{2}\delta} \leq A.$$

Write

$$\sum_5 = \sum_1^{[u^{-2}]} + \sum_{[u^{-2}]+1}^{\infty} = \sum_{5,1} + \sum_{5,2}.$$

Using (2.16), we have

$$\sum_{5,1} = O\left(u^{\alpha}\sum_1^{u^{-2}} n^{(\alpha-4)/2}\right) = O(1). \qquad \text{...(2.19)}$$

Using (2.17), we have

$$\sum_{5,2} = O\left(\sum_{[u^{-2}]+1}^{\infty} n^{(\alpha-4)/2}\right) + O\left(\sum_{[u^{-2}]+1}^{\infty} n^{(\alpha-2)/2}\rho^n(\delta)\right) + O\left(u^{\alpha}\sum_{[u^{-2}]+1}^{\infty} n^{(\alpha-2)/2}\rho^n(u)\right)$$

$$+O\left(u^{\alpha-2}\sum_{[u^{-2}]+1}^{\infty} n^{(\alpha-2)/2}\right) = O(1)$$

Collecting the above results, we see that $\sum_{n=1}^{\infty} | \alpha_n | < \infty$.

Now it remains to establish the absolute convergence of $\sum_{n=1}^{\infty} \beta_n$.

For this, we need only to prove

$$\sum_{n=1}^{\infty} \int_{\delta}^{\pi} | \phi(t) | \left| \frac{d}{dt} g_s(n,t,-1-\tfrac{1}{2}\alpha) \right| dt < \infty. \qquad \ldots(2.20)$$

Since $\frac{d}{dt} g_s(n,t,-1-\frac{1}{2}\alpha) = O(t^{-1} n^{-1-\alpha/2})+O(n^{(-2-\alpha)/2}\rho^{n}(t))$, the left hand side of (2.20) is less than

$$A\int_{\delta}^{\pi} | \phi(t) | \left(\sum_{n=1}^{\infty} t^{-1}n^{(-2-\alpha)/2} + \sum_{n=1}^{\infty} \rho^{n}(t)n^{-\alpha/2} \right)dt$$

$$\leqq A\left[\int_{\delta}^{\pi} \frac{| \phi(t) |}{t} dt + \int_{\delta}^{\pi} \frac{| \phi(t) | \, dt}{[1-\rho(t)]^{(2-\alpha)/2}}\right]$$

$$\leqq A\left[\int_{\delta}^{\pi} \frac{| \phi(t) |}{t} dt + \int_{\delta}^{\pi} \frac{| \phi(t) |}{t^{2-\alpha}} dt \right] \leqq A.$$

This terminates the proof of the theorem for the case $0<\alpha<2$.

(iii) We now proceed to prove the theorem for the case $\alpha=2$. Integrating by parts, we obtain

$$\frac{A_n(x)}{n} = \frac{2}{\pi}\left\{\Phi_1(\pi)-\pi\phi_2(\pi)\right\} \frac{\cos n\pi}{n} - \frac{1}{\pi}\int_0^{t}\left(t^2 \sin nt + \frac{2t \cos nt}{n}\right)d\phi_2(t)$$

$$+ \frac{2}{\pi}\int_0^{\pi}\frac{\sin nt}{n^2} d\phi_2(t) = \frac{2}{\pi} \left\{\Phi_1(\pi)-\pi\phi_2(\pi)\right\}\alpha'_n - \frac{1}{\pi}\beta'_n + \frac{2}{\pi}\gamma'_n, \text{ say.} \qquad \ldots(2.21)$$

Since $\phi_2(t)\in BV(0,\pi)$, the series $\sum_{n=1}^{\infty} \gamma'_n$ is absolutely convergent and hence a fortiori summable $| E,q |$, $q>0$. Using the notation introduced in (i) we notice that for $| E,q |$, $q>0$ summability of $\sum_{n=1}^{\infty} \beta'_n$ the following relation should hold.

$$\int_0^{\pi} | d\phi_2(t) | \sum_{n=1}^{\infty} | t^2 g_s(n,t,0)+tg_c(n,t,-1) | < \infty. \qquad \ldots(2\ 22)$$

Since $\int_0^{\pi} | d\phi_2(t) |$ is finite, it remains to show that unformly in $0<t\leqq\pi$

$$J = \sum_{n=1}^{\infty} | t^2 g_s(n,t,0)+tg_c(n,t,-1) | =O(1).$$

Using (2.1) and (2.2), we have

$$J=O\left(t^2 \sum_{n=1}^{\infty} \rho^n(t)\right)+O\left(t\sum_{n=1}^{\infty} t^{-1}n^{-2}\right)+O\left(t\sum_{n=1}^{\infty}\frac{\rho^n}{n}(t)\right)$$

$$=O(t^2/1-\rho(t))+O(1)+O\left(t \log \frac{1}{1-\rho(t)}\right)=O(1),$$

since $\rho^2(t)=1-\left(\frac{2\sqrt{q}}{1+q}\right)^2 \sin^2\frac{1}{2}t$.

In § 1, we notice that $|E_\alpha| \sim |E, \frac{\alpha}{1-\alpha}$, $0<\alpha<1$ and hence we can apply Definition 2 for summability $|E,q|$, $q>0$ of $\Sigma x'_n$. The series $\Sigma x'_n$ is summable $|E_\alpha|$, $0<\alpha<1$, $\sim |E,q|$, $q>0$ $\left(q=\frac{\alpha}{1-\alpha}\right)$, if

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{\nu=1}^{n}\binom{n}{\nu}\alpha^{\nu}(1-\alpha)^{n-\nu}\cos\nu\pi\right|<\infty. \qquad \ldots(2.23)$$

By formal calculation, it is seen that

$$\sum_{\nu=1}^{n}\binom{n}{\nu}\alpha^{\nu}(1-\alpha)^{n-\nu}\cos\nu\pi=(1-2\alpha)^n-(1-\alpha)^n, \text{ and}$$

hence (2.23) holds.

Thus Theorem 1 is proved completely.

§ 3. *Proof of Theorem* 2—The following lemma is to be used to prove Theorem.

Lemma (Bosanquet and Kestelmen 1939)—Suppose that $f_n(x)$ is measurable in (a,b) where $b-a<\infty$, for $n=1,2,\ldots$. Then a necessary and sufficient condition that for every function $\lambda(x)\in L(a,b)$ the function $f_n(x)\,\lambda(x)$ should be $L(a,b)$ and

$$\sum_{n=1}^{\infty}\left|\int_a^b \lambda(x)\, f_n(x)\,dx\right| \leqq K$$

is that

$$\sum_{n=1}^{\infty}|f_n(x)| \leqq K,$$

where K is an absolute constant, for almost every $x\in(a,b)$.

Assuming, as we may without loss in generality, that $\phi_1(t)\in AC$ i.e. $\phi(t)$ is absolutely continuous in $(0,\pi)$ and integrating by parts, we have

$$\frac{\pi}{2}\frac{A_n(x)}{n^{1/2-\epsilon}}=\frac{1}{n^{\frac{1}{2}-\epsilon}}\int_0^{\pi}\phi_1'(t)\,t\cos nt\,dt-\frac{1}{n^{3/2-\epsilon}}\int_0^{\pi}\phi_1'(t)\sin nt\,dt=p_n-q_n, \text{ say.}$$

We assume that $0<\epsilon<\frac{1}{2}$. This assumption does not impair the generality of the problem. Thus whenever $\phi_1(t)\in AC(0,\pi)$ the series $\Sigma\,|q_n|<\infty$ and hence a fortiori summable $|E,q|$, $q>0$. Using the notation of § 2.1, we notice that the necessary and sufficient condition for the summability $|E,q|$, $q>0$ of Σp_n for every function $f(t)$ such that $\phi_1(t)\in AC(0,\pi)$ is that

$$\sum_{n=1}^{\infty}\left|\int_0^{\pi}\phi(t)\, g_c(n,t,-\tfrac{1}{2}+\epsilon)\, dt\right| < \infty. \qquad \ldots(3.1)$$

Thus, necessary and sufficient condition for summability $|E,q|$, $q>0$ of Σp_n whenever $\phi_1(t)\in AC(0,\pi)$ is the same as the necessary and sufficient condition that whenever $\phi_1'(t)\in L(0,\pi)$ the condition (3.1) should hold. And this, by lemma is given by

$$\text{ess } \overline{bd}\, \Sigma\, |g_c(n,t,-\tfrac{1}{2}+\epsilon)| \leqq K \text{ in } 0<t\leqq\pi. \qquad \ldots(3.2)$$

Thus to establish our theorem, it is sufficient to show that (3.2) is false, that is

$$t\sum_n \left|g_c(n,t,-\tfrac{1}{2}+\epsilon)\right| \qquad \ldots(3.3)$$

is unbounded. From the proof of the inequality (2.2), we notice that (merely replacing sine by cosine)

$$g_\epsilon(n,t,-\tfrac{1}{2}+\epsilon)=\sum_{\nu=1}^{n-1}\Delta(\nu^{-\frac{1}{2}+\epsilon})\sum_{k=1}^{\nu}u_k(n)\cos kt + n^{-\frac{1}{2}+\epsilon}\, g_c(n,t,0).$$

Writing $M=\left[\frac{n+1}{q+1}\right]$ and proceeding as in the proof of (2.2), we can write

$$\begin{aligned} g_c(n,t,-\tfrac{1}{2}+\epsilon) &= \left[\sum_{\nu=M+1}^{n-1}\Delta(\nu^{-\frac{1}{2}+\epsilon}) + n^{-\frac{1}{2}+\epsilon}\right] g_c(n,t,o) \\ &+ \sum_{\nu=1}^{M}\Delta(\nu^{-\frac{1}{2}+\epsilon})\sum_{k=1}^{\nu}u_k(n)\sin kt - \sum_{\nu=M+1}^{n-1}\Delta(\nu^{-\frac{1}{2}+\epsilon}) \\ &\sum_{k=\nu+1}^{n}u_k(n)\sin kt = (M+1)^{-\frac{1}{2}+\epsilon} g_c(n,t,o) \\ &+ \sum\nolimits_1 - \sum\nolimits_2, \text{ say.} \end{aligned}$$

$u_k(n)$ is known to be monotonic increasing for the sum Σ_1 and monotonic decreasing for the sum Σ_2 and hence using the technique as in the proof of (2.2), it can be seen that

$$\begin{matrix}\Sigma_1 \\ \Sigma_2\end{matrix} = O(t^{-1}n^{-\frac{2}{3}+\epsilon}). \qquad \ldots(3.4)$$

Thus we have

$$t\sum_{n=1}^{\infty}\left|g_c(n,t,-\tfrac{1}{2}+\epsilon)\right| \geqq t\sum_{n=1}^{\infty}(M+1)^{-\frac{1}{2}+\epsilon}\left|g_c(n,t,0)\right|$$

(*equation continued on p. 709*)

$$-t\sum_{n=1}^{\infty}\left|\sum_{1}(n)\right|-t\sum_{n=1}^{\infty}\left|\sum_{2}(n)\right|.$$

The last two sums are finite in view of (3.4) and hence it remains to show that

$$I=t\sum_{n=1}^{\infty}(M+1)^{-\frac{1}{2}+\epsilon}\left|g_c(n,t,0)\right| \qquad \ldots(3.5)$$

is unbounded.

Using the notation introduced in §2(i), we have

$$I=t\sum_{n=1}^{\infty}(M+1)^{-\frac{1}{2}+\epsilon}\rho^n(t)\left|\cos n\theta\right|\geqq t\sum_{n=1}^{\infty}\rho^n(t)\,n^{-\frac{1}{2}+\epsilon}\cos n\theta$$

$$\geqq t\sum_{n=1}^{\infty}\rho^n(t)\,n^{-\frac{1}{2}+\epsilon}\cos^2 n\theta=\frac{1}{2}t\sum_{n=1}^{\infty}\frac{\rho^n(t)}{n^{\frac{1}{2}-\epsilon}}-\frac{1}{2}t\sum_{n=1}^{\infty}\frac{\rho^n(t)\cos 2n\theta}{n^{\frac{1}{2}-t}}$$

Since $\rho^n(t)/n^{\frac{1}{2}-\epsilon}$ is a decreasing function of n for fixed t, we have

$$\left|\tfrac{1}{2}t\sum_{n=1}^{\infty}\frac{\rho^n(t)\cos 2n\theta}{n^{\frac{1}{2}-\epsilon}}\right|\leqq\frac{t\rho(t)}{2}\max_{1\leqq M,M'\leqq\infty}\left|\sum_{M}^{M'}\cos 2n\theta\right|$$

$$\leqq\frac{t\rho(t)}{2\sin\theta}\leqq\frac{t}{2\sin\theta}.$$

Since $\theta\sim\left(\frac{q}{1+q}\right)t$ for small t, we have $\sin\theta\sim\left(\frac{q}{1+q}\right)t$ as $t\to+0$, and thus

$$\sum_{n=1}^{\infty}\frac{\rho^n(t)}{n^{\frac{1}{2}-\epsilon}}\cos 2n\theta=O(1).$$

Again

$$\tfrac{1}{2}t\sum_{n=1}^{\infty}\frac{\rho^n(t)}{n^{\frac{1}{2}-\epsilon}}=\tfrac{1}{2}t\frac{\Gamma(\frac{1}{2}+\epsilon)}{[1-\rho(t)]^{\frac{1}{2}+\epsilon}}=\tfrac{1}{2}t\frac{\Gamma(\frac{1}{2}+\epsilon)[1+\rho(t)]^{\frac{1}{2}+\epsilon}}{[1-\rho^2(t)]^{\frac{1}{2}+\epsilon}}$$

$$\leqq\tfrac{1}{2}t\frac{(\frac{1}{2}+\epsilon)}{[1-\rho^2(t)]^{\frac{1}{2}+\epsilon}}=\tfrac{1}{2}t\frac{\Gamma(\frac{1}{2}+\epsilon)}{\left(\frac{2\sqrt{q}}{1+q}\sin\frac{1}{2}t\right)^{1+2\epsilon}}>\frac{\Gamma(\frac{1}{2}+\epsilon)}{2\left(\frac{\sqrt{q}}{1+q}\right)^{1+2\epsilon}}t^{-2\epsilon}$$

There fore a given positive number H however large, it is possible to find t such that

$$\tfrac{1}{2}t\sum_{n=1}^{\infty}\frac{n(t)}{n^{\frac{1}{2}-\epsilon}}>H.$$

This completes the proof of Theorem 2.

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