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MULTILINEAR GENERATING FUNCTIONS FOR JACOBI POLYNOMIALS AND FOR THEIR TWO-VARIABLE GENERALIZATIONS

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A multilinear generating function involving the products of several Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is given. The corresponding multilinear generating functions for class II, class III, class IV and class V type Koornwinder polynomials that are two-variable generalizations of Jacobi polynomials are obtained as applications.

1. INTRODUCTION

For some considerable time, generalizations of Mehler's formula for Hermite polynomials and the Hardy-Hille formula for the Laguerre polynomials have been well known. These extensions are in the form of multilinear generating functions. Such formulae are also known for $P_n^{(\alpha-n,\beta-n)}(x)$ or $P_n^{(-\alpha,\beta-n)}(x)$, the variants of the orthogonal Jacobi polynomials; see Srivastava and Singhal (1972) and Thakare (1980). The purpose of this paper is to obtain multilinear generating functions involving the product of several Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

As significant applications of our considerations we are led to multilinear generating functions for different two-variable generalizations of Jacobi polynomials introduced recently by Koornwinder (1975).

2. MAIN RESULT

In this section we shall obtain the following principal result of our paper:

$$\sum_{n_1,\ldots,n_k=0}^{\infty} (\delta)_{m+\Sigma n_i}(\lambda+m)_{\Sigma n_i}\ {}_2F_1(-m-\Sigma n_i,\ \lambda+m+\Sigma n_i;\ \delta;\ x)$$

$$\prod_{j=1}^{k}\frac{{}_2F_1(-n_j,-\beta_j-n_j;\alpha_j;y_j)u_j^{n_j}}{(1+\beta_j)_{n_j}\, n_j!} = (\delta)_m(1-x)^{-\lambda-m}$$

$$\times\ F_C^{(2k+1)}\Big(\delta+m,\ \lambda+m;\ \delta,\ \alpha_1,\ldots,\alpha_k,\ 1+\beta_1,\ldots,\ 1+\beta_k;\ \frac{x}{x-1},$$

$$\times\ \frac{y_1u_1}{1-x},\ldots,\frac{y_ku_k}{1-x},\frac{u_1}{1-x},\ldots,\frac{u_k}{1-x}\Big) \qquad \ldots(2.1)$$

where m is some fixed integer, δ and λ are complex parameters, $F_C^{(n)}$ is Lauricella's hypergeometric function of n variables of the third kind [see Exton 1976, p. 41].

PROOF: Consider the sum

$$S_0 = \sum_{n_1,\cdots,n_k=0}^{\infty} (\delta)_{m+\Sigma n_i}(\lambda+m)_{\Sigma n_i} \prod_{j=1}^{k} \frac{{}_2F_1(-n_j,-\beta_j-n_j;\alpha_j;y_j)\, u_j^{n_j}}{(1+\beta_j)_{n_j} n_j!}.$$

After some simplification, one can write

$$S_0 = \sum_{\substack{r_1,\cdots,n_k=0\\ r_1,\cdots,n_k=0}}^{\infty} \frac{(\delta)_{m+\Sigma n_i+\Sigma r_i}(\lambda+m)_{\Sigma n_i+\Sigma r_i}(y_1u_1)^{r_1}\cdots(y_ku_k)^{r_k}u_1^{n_1}\cdots u_k^{n_k}}{(\alpha_1)_{r_1}\cdots(\alpha_k)_{r_k}(1+\beta_1)_{n_1}\cdots(1+\beta_k)_{n_k}r_1!\cdots r_k!n_1!\cdots n_k!}$$

Thus finally we obtain

$$S_0 = (\delta)_m F_C^{(2k)}(\delta+m,\lambda+m;\alpha_1,\ldots,\alpha_k,1+\beta_1,\ldots,1+\beta_k;y_1u_1,\cdots,y_ku_k,u_1,\cdots,u_k)\ldots(2.2)$$

Let S denote the left-hand side of (2.1). By using the familiar Euler transformation (Rainville 1960, p. 60), one has

$$S=(1-x)^{-\lambda-m}\sum_{n_1,\cdots,n_k=0}^{\infty} (\delta)_{m+\Sigma n_i}(\lambda+m)_{\Sigma n_i}(1-x)^{-\Sigma n_i}$$

$$\times\, {}_2F_1\left(\delta+m+\Sigma n_i,\lambda+m+\Sigma n_i,\ \delta;\ \frac{x}{x-1}\right)\prod_{j=1}^{k}\frac{{}_2F(-n_j,-\beta_j-n_j;\alpha_j;y_j)\, u_j^{n_j}}{(1+\beta_j)_{n_j}\, n_j!}$$

$$=(1-x)^{-\lambda-m}\sum_{l=0}^{\infty}\left(\frac{x}{x-1}\right)^{l}\frac{(\lambda+m)_l}{l!}\sum_{n_1,\cdots,n_k=0}^{\infty}(\delta+l)_{m+\Sigma n_i}(\lambda+l+m)_{\Sigma n_i}$$

$$\prod_{j=1}^{k}\frac{{}_2F_1(-n_j,-\beta_j-n_j;\alpha_j;y_j)}{(1+\beta_j)_{n_j}\, n_j!}\left(\frac{u_j}{1-x}\right)^{n_j}.$$

Now using (2.2) and writing the expansion of the Lauricella function involved, we obtain

$$S=(1-x)^{-\lambda-m}\sum_{\substack{l,r_1,\cdots,r_k=0\\ m_1,\cdots,m_k=0}}^{\infty}\frac{(x/x-1)^l\,(\lambda+m)_l\,(\delta+l)_m(\delta+m+l)_{\Sigma r_i+\Sigma m_i}}{l!\,(\alpha_1)_{r_1}\cdots(\alpha_k)_{r_k}\,(1+\beta_1)_{m_1}\cdots(1+\beta_k)_{m_k}}$$

$$\times\ \frac{(\lambda+m+l)_{\Sigma r_i+\Sigma m_i}}{r_1!\cdots r_k!\,m_1!\cdots m_k!}\left(\frac{y_1u_1}{1-x}\right)^{r_1}\cdots\left(\frac{y_ku_k}{1-x}\right)^{r_k}\left(\frac{u_1}{1-x}\right)^{m_1}\cdots\left(\frac{u_k}{1-x}\right)^{m_k}.$$

This can be further simplified and can be put in the desired form (2.1).

3. APPLICATIONS

From (2.1), in view of the relationships [Rainville 1960, p. 254, eqns. (1) and (2)], we have

$$\sum_{n_1,\ldots,n_k=0}^{\infty} (m+\Sigma n_i)!(1+\alpha+\beta+m)_{\Sigma n_i} P_{m+\Sigma n_i}^{(\alpha,\beta)}(x) \prod_{j=1}^{k} \frac{P_{n_j}^{(\alpha_j,\beta_j)}(y_j)u_j^{n_j}}{(1+\alpha_j)_{n_j}(1+\beta_j)_{n_j}}$$

$$=(1+\alpha)_m\left(\frac{1+x}{2}\right)^{-1-\alpha-\beta-m} F_C^{(2k+1)}\Big(1+\alpha+m,\ 1+\alpha+\beta+m;\ 1+\alpha,\ 1+\alpha_1,\ldots,$$

$$1+\alpha_k, 1+\beta_1,\ldots,1+\beta_k;\ \frac{x-1}{x+1},\frac{(y_1-1)u_1}{x+1},\ldots,\frac{(y_k-1)u_k}{x+1},\frac{(y_1+1)u_1}{1+x},\ldots,\frac{(y_k+1)u_k}{1+x}\Big). \quad \ldots(3.1)$$

For $k=1$, (3.1) yields a result of Manocha and Sharma (1967).

In (2.1), put $\beta_1=\ldots=\beta_k=\lambda$, replace x by x/λ, each y_j by $-y_j/\lambda$, δ by $1+\alpha$, each α_j by $1+\alpha_j$ and take limit as $\lambda\to\infty$. Then, due to relationship [Rainville 1960, p. 200, eqn. (1)] and in view of the result due to Srivastava [1971, p. 4, eqn. (12)]:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\ldots+m_n)\frac{x_n^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}=\sum_{N=0}^{\infty} f(N)\,\frac{(x_1+\ldots+x_n)^N}{N!}, \quad \ldots(3.2)$$

one obtains the known multilinear generating function of Srivastava and Singhal (1972, p. 1239, eqn. (12)) for the classical Laguerre polynomials. This result can be derived from (3.1) by using (3.2) and the familiar relationship

$$L_n^{\alpha}(x)=\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2x\beta^{-1}).$$

4. Further Applications

Recently, Koornwinder (1975) defined seven different types of orthogonal polynomials in two variables which are generalizations of the Jacobi polynomials. We shall obtain multilinear generating function for the following Koornwinder's two-variable analogues of Jacobi polynomials.

Class II—For $\alpha>-1$ the polynomials given below are orthogonal with respect to the weight function $(1-x^2-y^2)^{\alpha}$ on the unit disc:

$${}_2P_{n,k}^{\alpha}(x,y)=P_{n-k}^{(\alpha+k+(1/2),\alpha+k+(1/2))}(x)\,(1-x^2)^{k/2}\,P_k^{(\alpha,\alpha)}\left(\frac{y}{(1-x^2)^{1/2}}\right),\quad n\geqslant k\geqslant 0. \quad \ldots(4.1)$$

Class III—For $\alpha,\beta>-1$, the following polynomials are orthogonal with respect to the weight function $(1-x)^{\alpha}(x-y^2)^{\beta}$ on the region $\{(x,y)\mid y^2<x<1\}$ which is bounded by a straight line and a parabola:

$${}_3P_{n,k}^{\alpha,\beta}(x,y)=P_{n-k}^{(\alpha,\beta+k+(1/2))}(2x-1)\,x^{k/2}\,P_k^{(\beta,\beta)}(y/\sqrt{x}),\quad n\geqslant k\geqslant 0. \quad \ldots(4.2)$$

Class IV—For $\alpha,\ \beta,\ \nu>-1$ the polynomials defined below are orthogonal with respect to the weight function $(1-x)^{\alpha}(x-y)^{\beta}y^{\nu}$ on the triangular region $\{(x,y)\mid 0<y<x<1\}$.

$${}_4P_{n,k}^{\alpha,\beta,\nu}(x,y)=P_{n-k}^{(\alpha,\beta+\nu+2k+1)}(2x-1)\,x^k\,P_k^{(\beta,\nu)}\left(\frac{2y}{x}-1\right),\quad n\geqslant k\geqslant 0. \quad \ldots(4.3)$$

Class V—For $\alpha,\ \beta,\ \nu,\ \delta>-1$ the polynomials

$${}_5P_{n,k}^{\alpha,\beta,\nu,\delta}(x,y)=P_{n-k}^{(\alpha,\beta)}(x)\,P_k^{(\nu,\delta)}(y),\quad n\geqslant k\geqslant 0. \quad \ldots(4.4)$$

are orthogonal with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\nu}(1+y)^{\delta}$ on the square $\{(x,y) \mid -1<x<1,\ -1<y<1\}$.

For our analysis we also need the following result of Srivastava and Singhal [(1972), p. 1246, eqn. (30)]:

$$\sum_{n_1,\dots,n_k=0}^{\infty} (m+\Sigma n_i)!\, P_{m+\Sigma n_j}^{(\alpha-m-\Sigma n_j,\beta-m-\Sigma n_i)}(x) \prod_{j=1}^{k} \frac{P_{n_j}^{(\beta_j-n_j,\ \nu_j-n_j)}(y_j)\, u_j^{n_j}}{(-\beta_j-\nu_j)_{n_j}}$$

$$=(-1)^m(-\alpha-\beta)_m\left(\frac{x+1}{2}\right)^{\alpha}\left(\frac{x-1}{2}\right)^{m-\alpha}\Omega_k^{\alpha+\beta-m}\, F_A^{(k+1)}\Big(m-\alpha-\beta,-\alpha,-\beta_1,\dots,$$

$$-\beta_k;-\alpha-\beta,-\beta_1-\nu_1,\dots,-\beta_k-\nu_k;\frac{2}{(x+1)\Omega_k},\frac{(x-1)u_1}{2\Omega_k},\dots,\frac{(x-1)u_k}{2\Omega_k}\Big) \quad \dots(4.5)$$

where $\Omega_k=1-\frac{x-1}{4}\sum_{j=1}^{k}(y_i-1)u_i$, and $F_A^{(n)}$ is Lauricella's hypergeometric function of n variables of the first kind (see Exton 1976, p. 41).

From the definitions (4.1) to (4.4) and the relations (3.1), (4.5), we obtain after appropriately regrouping the terms involved the following multilateral generating functions for two-variable analogues of Jacobi polynomials- In what follows M stands for $(m+\sum_1^r n_i)!\,(n+\sum_1^r m_i)!$.

$$\sum_{\substack{n_1,\dots,n_r=0\\ m_1,\dots,m_r=0}}^{\infty} M(2+2\alpha+m)_{\Sigma n_i}\ {}_2P_{m+n+\Sigma n_i+\Sigma n_i,\ n+\Sigma m_i}^{\alpha-n-\Sigma m_i}(x,y)$$

$$\times\prod_{j=1}^{r}{}_2P_{n_j+m_j,\ m_j}^{\alpha_j-m_j}(x_j,y_j)\, u_j^{n_j} t_j^{m_j} \Big/ \left[\left(\frac{3}{2}+\alpha_j\right)_{n_j}\right]^2(-2\alpha_j)_{m_j}$$

$$=\left(\frac{3}{2}+\alpha\right)_m(-2\alpha)_n\left(\frac{-1}{2}\right)^n\left(\frac{1+x}{2}\right)^{-2-2\alpha-m}(y+\sqrt{1+x^2})^{\alpha}(y-\sqrt{1-x^2})^{n-\alpha}\theta_r^{2\alpha-n}$$

$$F_C^{(2r+1)}\Big(\tfrac{3}{2}+\alpha+m,2+2\alpha+m;\tfrac{3}{2}+\alpha,\tfrac{3}{2}+\alpha_1,\dots,\tfrac{3}{2}+\alpha_r,\tfrac{3}{2}+\alpha_1,\dots,\tfrac{3}{2}+\alpha_r;\frac{x-1}{x+1},$$

$$\frac{(x_1-1)u_1}{1+x},\dots,\frac{(x_r-1)u_r}{1+x}\frac{(x_1+1)u_1}{1+x},\dots,\frac{(x_r+1)u_r}{1+x}\Big)F_A^{(r+1)}(n-2\alpha,-\alpha,-\alpha_1,\dots,-\alpha_r;$$

$$-2\alpha,-2\alpha_1,\dots,-2\alpha_r;\frac{2\sqrt{1-x^2}}{[y+\sqrt{1-x^2}]\theta_r},\frac{(y-\sqrt{1-x^2})\sqrt{1-x_1^2}\,t_1}{2\theta_r},\dots,$$

$$\frac{(y-\sqrt{1-x})\sqrt{1-x_r^2}\,t_r}{2\theta_r} \quad \dots(4.6)$$

where $\theta_r=1-\frac{y-\sqrt{1-x^2}}{4}\sum_{i=1}^{\infty}(y_i-\sqrt{1-x_i^2})t_i$.

(*equation continued. on p.* 715)

$$\sum_{\substack{n_1,\dots,n_r=0\\ m_1,\dots,m_r=0}}^{\infty} M\left(\frac{3}{2}+\alpha+\beta+m\right)_{\Sigma n_i}\ {}_3P^{\alpha,\beta-n-\Sigma m_i}_{m+n+\Sigma n_i+\Sigma m_i,\ n+\Sigma m_i}(x,y)$$

$$\times \prod_{j=1}^{r} {}_3P^{\alpha_j,\beta_j-m_j}_{n_j+m_j,m_j}(x_j,y_j)\ {u_j}^{n_j}\ {t_j}^{m_j} \Big/ (1+\alpha_j)_{n_j}\left(\tfrac{3}{2}+\beta_j\right)_{n_j}(-2\beta_j)_{m_j}$$

$$=(-\tfrac{1}{2})^n(1+\alpha)_m(-2\beta)_n x^{-\alpha-\beta-m-(3/2)}(y+\sqrt{x})^{\beta}(y-\sqrt{x})^{n-\beta}\Delta_r^{2\beta-n}\ F_C^{(2r+1)}$$

$$\times\left(1+\alpha+m,\tfrac{3}{2}+\alpha+\beta+m;1+\alpha,1+\alpha_1,\dots,1+\alpha_r;\tfrac{3}{2}+\beta_1,\dots,\tfrac{3}{2}+\beta_r;\frac{x-1}{x},\frac{(x_1-1)u_1}{x},\dots,\right.$$

$$\times\left.\frac{(x_r-1)u_r}{x},\frac{x_1u_1}{x},\dots,\frac{x_ru_r}{x}\right)F_A^{(r+1)}\ (n-2\beta,\ -\beta,\ \beta_1,\dots,-\beta_r;-2\beta,-2\beta_1,\dots,$$

$$\left.-2\beta_r,\ \frac{2\sqrt{x}}{(y+\sqrt{x})\Delta_r},\frac{\sqrt{x_1}(y-\sqrt{x})t_1}{2\Delta_r},\dots,\frac{\sqrt{x_r}(y-\sqrt{x})t_r}{2\Delta_r}\right) \qquad \dots(4.7)$$

where $\Delta_r= 1-\dfrac{y-\sqrt{x}}{4}\displaystyle\sum_{i=1}^{r}(y_i-\sqrt{x_i})t_i.$

$$\sum_{\substack{n_1,\dots,n_r=0\\ m_r,\dots m_1=0}}^{\infty} M(2+\alpha+\beta+\nu+m)_{\Sigma n_i}\ {}_4P^{\alpha,\beta-n-\Sigma m_i,\ \nu-n-\Sigma m_i}_{m+n+\Sigma n_i+\Sigma m_i,\ n+\Sigma m_i}(x,y)$$

$$\times\prod_{j=1}^{r}\frac{{}_4P^{\alpha_j,\beta_j,-m_j,\ \nu_j-m_j}_{n_j+m_j,\ m_j}(x_j,y_j)\ {u_j}^{n_j}\ {t_j}^{m_j}}{(1+\alpha_j)_{n_j}\ (2+\beta_j+\nu_j)_{n_j}(-\beta_j-\nu_j)_{m_j}}$$

$$=(-1)^n(1+\alpha)_m(-\beta-\nu)_m x^{-2-\alpha-\beta-\nu-m}\ y^{\beta}(y-x)^{n-\beta}\ \mu_r^{\beta+\nu-n}\ F_C^{(2r+1)}\ (1+\alpha+m,$$

$$2+\alpha+\beta+\nu+m;\ 1+\alpha,\ 1+\alpha_1,\dots 1+\alpha_r,\ 2+\beta_1+\nu_1,\dots,2+\beta_r+\nu_r;\ (x-1)x^{-1},$$

$$(x_1-1)u_1x^{-1},\dots,\ (x_r-1)u_rx^{-1},\ x_1u_1x^{-1},\dots,x_ru_rx^{-1}).\ F_A^{(r+1)}\ (n-\beta-\nu,-\beta,=\beta_1,\dots,$$

$$-\beta_r;\ -\beta-\nu,\ -\beta_1-\nu_1,\dots,-\beta_r-\nu_r;\ x(y\mu_r)^{-1},\ (y-x)x_1t_1\mu_r^{-1},\dots,(y-x)x_rt_r\mu_r^{-1}) \qquad \dots(4.8)$$

where $\mu_r=1-(y-x)\displaystyle\sum_{i=1}^{r}(y_i-x_i)t_i.$

$$\sum_{\substack{n_1,\dots,n_r=0\\ m_1,\dots,m_r=0}}^{\infty} M(1+\alpha+\beta+m)_{\Sigma n_i}(1+\nu+\delta+n)_{\Sigma m_i}\ {}_5P^{\alpha,\beta,\nu,\delta}_{m+n+\Sigma n_i+\Sigma m_i,\ n+\Sigma m_i}(x,y)$$

(*equation continued on p.* 716)

$$\times \prod_{j=1}^{r} P_{n_j+m_j,m_j}^{\alpha_j,\beta_j,\nu_j,\delta_j}(x_j y_j) \frac{u_j^{n_j} t_j^{m_j}}{(1+\alpha_j)_{n_j}(1+\beta_j)_{n_j}(1+\nu_j)_{m_j}(1+\delta_j)_{m_j}}$$

$$= (1+\alpha)_m(1+\nu)_n \left(\frac{1+x}{2}\right)^{-1-\alpha-\beta-m} \left(\frac{1+y}{2}\right)^{-1-\nu-\delta-n} F_C^{(2r+1)}(1+\alpha+m,$$
$$1+\alpha+\beta+m;\ 1+\alpha,\ 1+\alpha_1,\ldots,1+\alpha_r,\ 1+\beta_1,\ldots,1+\beta_r;\ (x-1)X,(x_1-1)u_1X,\ldots,$$
$$(x_r-1)u_rX,\ (x_1+1)u_1X,\ldots,(x_r+1)u_rX)F_C^{(2r+1)}\ (1+\nu+n,\ 1+\nu+\delta+n;\ 1+\nu,\ 1+\nu_1$$
$$,\ldots,1+\nu_r,\ 1+\delta_1,\ldots,1+\delta_r;\ (y-1)Y,\ (y_1-1)t_1Y,\ldots,(y_r-1)t_rY,\ (y_1+1)t_rY,\ldots,$$
$$(y_r+1)t_rY) \qquad \ldots(4.9)$$

where $X=1/(1+x)$ and $Y=1/(1+y)$ on the right-hand side.

5. Concluding Remarks

There are known two-variable generalizations of the Laguerre polynomials which can be expressed as products of two Laguerre polynomials as indicated by Koornwinder (1975). Thus multilinear generating functions for such two-variable analogues of the Laguerre polynomials can easily be obtained.

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