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INFLUENCE OF INTERNAL STRESSES ON MOTION OF A BODY*

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The Newtonian motion of a two-dimensional body on a smooth surface is studied in this paper. The body consists of 12 elastic bands of negligible thickness. By changing the length of the bands, one can create or change the forces in the bands. It is found that these internal forces will influence the motion. By comparing this particular example with the study of general relativity, one is able to formulate some conjectures regarding the role of stresses on motion in the context of general relativity.

1. INTRODUCTION

In Newtonian mechanics (N.M.) only mass is the source of the gravitational field (i.e., the Newtonian potential) and it is only mass upon which this field is acting. The Einstein equation of general relativity (G.R.), on the other hand, put stresses on an equal footing with mass; we therefore expect that the gravitational field (i.e., the metric) and the motion of bodies is influenced by stresses as well. However, not much is known in this area; in fact, no exact solution of the Einstein equations corresponding to two isolated bodies with positive mass is known. One reason for this is the complexity of the Einstein equations, another is our lack of intuition. If we could live in strongly curved space-time, say, on an extremely massive planet, we would frequently observe consequences of strong curvature. If, for instance, we discovered experimentally how "pulling wires and tightening screws" inside a spacecraft influences its orbit, the search for corresponding solutions of the Einstein equations would be easier. Failing this, the next best thing is to work with models—in theory or experiment—of which at least some features are shared by the physical reality we want to study. Intuition thus developed will lead us to conjectures which, hopefully, are correct. An example for this is the use of space-time diagrams in G.R.

The model which we study in sections 12-5 belongs to N.M. and no elements extraneous to this theory are introduced. The manner in which this model might help us to understand some features of G.R. is discussed in sections 16/8.

2. THE BODY

In N.M. we frequently study bodies with less than three dimensions; point-particles, membranes and shells are examples. Here we study the motion of a two-dimensional body in a smooth and frictionless surface. Our sole aim is to see how

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stresses in the body influence its motion and we therefore neglect any gravitational field which may be present.

The body consists of 12 straight elastic bars of negligible thickness (i.e., bands) connected by seven frictionless joints as depicted in Fig. 1.

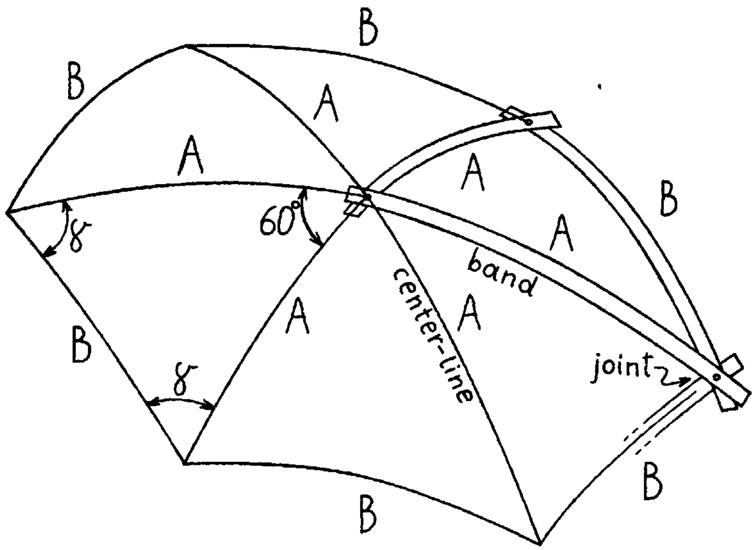

Fig. 1. The body on a sphere; A and B are the actual length of an inner and outer band respectively; a and b are the corresponding length in the unstrained state.

In an experiment, the elastic bars could be made from, steel bands similar to those used in shipping of heavy boxes. The surface might consist of two lubricated and, in order to make observations easy, transparent surfaces, separated by a gap just wide enough to accommodate the body.

3. The Body on a Sphere

An isolated band, placed on a sphere of radius R, will have no strain energy, since the resistance against bending in the direction perpendicular to the band is negligible. Let us calculate the strain energy which is stored in a single band if we exert on the two ends a pull or push of equal magnitude and tangent to the band. The centre line of the band will coincide with a great circle on the sphere because of symmetry (if we prevent buckling). The force in the band will be tangent to the centre line, and the magnitude of the force is constant (a consequence of the theory of chains and cables). The band is made of elastic material. Hooke's law therefore gives

$$\Delta l \equiv L - l = lF/DY \equiv lF/c \qquad \dots(1)$$

where l = length of band in unstrained state,

L = length of band in strained state,

F = force in band, negative if compressive,

Δl = change in length of band due to F,

D = surface area of cross section of band,

Y = modulus of elasticity

$c = DY.$

Such an elastic band of length l thus behaves like a spring with spring constant

$$k = l/DY = l/c \qquad \ldots(2)$$

The strain energy S_l stored in the band is equal to the potential energy of the "spring"; i.e.,

$$S_l = \tfrac{1}{2}\,\Delta l F = lF^2/2c = (L - l)^2 c/2l \qquad \ldots(3)$$

Equations (1), (2) and (3) are the same as for a band or spring in a plane since the radius R does not appear.

If we connect the 12 bands by frictionless joints, then the bending moments in the bands at the joints are zero by definition. The force in each band is tangent to the centre line, as assumed for the single band considered above. If in an actual model the bands are connected by bolts, these should not be tightened completely. To link the first 11 bands will require no energy. However, on a surface such as a plane, the last band will generally not fit (we have not assumed that all bands have the same length). Energy will be needed to force joints apart so that we can fit in the last band. This energy will appear as strain energy. The fact that 11 but not 12 bands on a surface can be linked without effort, results from the body being an example of a plane over-rigid truss (Shames 1958), overdetermined of degree 1; 11 of the bands thus from a just rigid truss.

It is important to understand that the forces in the 12 bands are generally non-zero although no external forces are applied. Bodies which have stresses even when no external forces are acting are called prestressed or self-straining. A conceptually simple example of self-straining body is given by a torus made of steel, after a slice is removed, and the two cuts thus created are forced and welded together. To produce examples of bodies with large self-stress, one could let molten glass drip into water. The glass drops thus obtained are strong enough to withstand hammer blows, but they turn to dust if their tip is broken off Overrigid trusses are generally self-straining, whereas just rigid trusses never are.

Let us now assume that there is a mechanism which allows flatlanders riding the body to change the (unstrained) length of at least one band. A way to realize this is to provide one band on one end with several closely spaced holes which allow us to move the bolt forming the joint. Greater mathematical simplicity is achieved, however, if we assume that all six inner bands have, if unstrained, the same but variable length a; whereas the outer six bands have, if unstrained, the same constant length b. The actual lengths of the inner and outer bands are denoted by A and B, respectively. Note that the actual lengths, in the presence of forces, are unequal to the unstrained lengths. On a sphere all outer (respectively inner) bands experience the same force because of symmetry of the problem and therefore have the same length B (respectively A) in the strained state.

For the equations of motion we need the strain energy S of the body. It is equal to the sum of the strain energies of the 12 bands. With eqn. (3) we find

$$S = \frac{6(A-a)^2c}{2a} + \frac{6(B-b)^2c}{2b}. \quad \text{..(4)}$$

We assume that a, b, c and also R are given; $A \equiv \alpha R$ and $B \equiv \beta R$ can then be found from the following set of equations:

$$\sin\gamma = \cos 30° (\cos \alpha/2)^{-1}, \quad \text{...(5)}$$

$$\cos\beta = \cos^2\alpha + \sin^2\alpha \cos 60° = 1 - \tfrac{1}{2}\sin^2\alpha, \quad \text{...(6)}$$

$$0 = 2\cos\gamma\,(B-b)\,b^{-1} + (A-a)\,a^{-1}, \quad \text{...(7)}$$

where γ is the angle between an outer and an inner band. Equations (5) and (6) are well known from spherical trigonometry. Equation (7) states that the sum of all forces on an outer joint must be equal. This is trivially satisfied for the component of the force perpendicular to the inner band, therefore only the equation for the component parallel to the inner band appears. We have assumed that all bands are made of the same elastic material and have the same cross section. We then have invariance of the body under a reflection and under rotations by 60°. n (n = 1, 2. 3...,) and it suffices to consider one outer joint only. Equations (5). (6) and (7) cannot be solved exactly. We therefore assume that the body is much smaller than the sphere:

$$0 < a, b \ll R \Rightarrow 0 < \alpha, \beta \ll 1. \quad \text{...(8)}$$

This allows us to neglect higher-order terms in the Taylor expansions of sin α, cos $\alpha/2$ and cos β. We find

$$\sin\gamma \approx \tfrac{1}{2}\sqrt{3}(1 + (\alpha^2/8)) \Rightarrow \cos\gamma \approx (\tfrac{1}{2} - (3\alpha^2/16)). \quad (5')$$

The assumption (8) together with (6) also implies that

$$\beta \approx \alpha - (\alpha^3/8). \quad (6')$$

Substituting eqns. (5′) and (6′) into eqn. (7) gives

$$0 = 8R\alpha - R\alpha^3 - 8b - 3R\alpha^3 + (\tfrac{3}{8})R\alpha^5 + 3b\alpha^2 - 8b + 8R\alpha ba^{-1}. \quad \text{...(9)}$$

Neglecting the term with α^5 gives a cubic equation for α with three real roots. The only root satisfying the inequality (8) is given by

$$A = R\alpha \approx (4/m)\,[1 - 3\,R^{-2}m^{-2} + 16\,R^{-2}b^{-1}m^{-3}] \quad \text{...(10)}$$

where m is the harmonic mean of a and b :

$$m \equiv 2a^{-1} + 2b^{-1} = 2(a+b)/ab. \quad \text{...(11)}$$

Equation (6) now gives

$$B = R\beta \approx (4/m)\,[1 - 5R^{-2}m^{-2} + 16\,R^{-2}b^{-1}m^{-3}]. \quad \text{...(12)}$$

4. The Body on a Surface of Constant Curvature

The expression for A and B which we derived for a sphere of radius R depend on the intrinsic geometry of the surface only, and are therefore equally valid for any surface having constant and nonnegative Gaussian curvature $K = R^{-2}$. Furthermore, the equations which we get by substituting K for R^{-2} in eqns. (10) and (12) are equally valid if the curvature is negative. We thus have

$$A = (4/m)\,[1 - 3Km^{-2} + 16Kb^{-1}m^{-3}] \quad \text{..(10')}$$

$$B = (4/m)\,[1 - 5Km^{-2} + 16Kb^{-1}m^{-3}] \quad \text{...(12')}$$

for any surface with constant Gaussian curvature K, a and b satisfying

$$0 < a, b \ll |K|^{-1/2}. \quad (8')$$

Using eqns. (10′), (12) and (4) we find, for the strain-energy S of the body,

$$S = 3c\ [(a+b)^{-1}(a-b)^2 - 2K(ab)^2(a+b)^{-4}(a-b) + \text{terms of order } K^2 \text{ and higher order}]. \quad ...(13)$$

Because a factor $(a-b)$ appears in the first two terms, we can ignore the term of order K^2 only if

$$| a - b | \gg | K |^{-1/2}. \quad ...(14)$$

If this is not the case, we have to repeat the calculations leading from eqn (5) to Eqn. (13), keeping more terms in the Taylor expansions. But we will find in either case that S depends only on a, b, c and K

5. The Equations of Motion

We now drop the assumption that the surface has constant curvature K; instead we assume that K is changing slowly, so that for the calculation of the strain-energy S, K can be assumed to be constant in the domain occupied by the body at a certain instant. We thus can use expression (13) for S, taking for K the Gaussian curvature at the position of the centre joint. We shall express the fact that both a and K are variable, whereas b and c are constants, by writing

$$S = S(a,K). \quad ...(15)$$

The body cannot move as a rigid body; different particles, therefore, will have slightly different velocities. Let us for simplicity assume that the mass m of the body—the payload—is concentrated in its centre and we therefore can take the velocity of the centre joint as the velocity of the body. Curvilinear coordinates on the surface are denoted by u^1, u^2. The centre of the body will trace out a curve $u^1(t)$, $u^2(t)$ where t is the time and the components of the velocity veetor are

$$\dot{u}^1 = du^1/dt, \ \dot{u}^2 = du^2/dt. \quad ...(16)$$

The kinetic energy of the body is then given by (Synge and Schild 1964, pp. 151, 168)

$$T = \frac{1}{2} m \sum_{i=1}^{2} \sum_{k=1}^{2} a_{ik}\dot{u}^i\dot{u}^k, \quad ...(17)$$

where $a_{ik}(u^1,u^2)$ is the metric tensor of the surface. If the surface is given in the form

$$z = f(x, y),$$

a_{ik} and K can be found readily (Eisenhart 1964). The equations of motion for the body are (Synge and Schild 1964, p. 152)

$$\frac{d(\partial T/\partial \dot{u}^n)}{dt} - \frac{\partial T}{\partial u^n} = U_n, \quad n = 1,2. \quad ...(18)$$

The generalized force U_n has to be found from

$$W = \sum_{n=1}^{2} U_n du^n, \quad ...(19)$$

where W is a 1-form representing the work done in an arbitrary infinitesimal displacement (du^1, du^2) (Shen 1976),

Because of assumption (8′) there exist, at least in some domain, quasi-Cartesian coordinates x^1, x^2. These are coordinates in which the metric tensor has the form

$$a_{ik}(x^1,x^2)\approx\begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k. \quad i,k = 1,2. \end{cases} \qquad \ldots(20)$$

In these coordinates eqns. (17) and (18) go over into more familiar form

$$T\approx\tfrac{1}{2}m\,[(\dot{x}^1)^2 + (\dot{x}^2)^2], \qquad \ldots(17')$$

$$m\ddot{x}\approx\mathscr{X}_n \qquad \ldots(18')$$

where $\mathscr{X}_n$ has to be found from

$$W = \sum_{n=1}^{2} \mathscr{X}_n dx^n \qquad \ldots(19')$$

We now try to find U_n. Let E denote the energy contained in a battery of the body. This battery, we assume, stores/provides the energy generated/consumed by a change of length a. Any work done is done by this battery or the strains; we thus have

$$W = - dS - dE. \qquad \ldots(20)$$

From eqn. (15) we find

$$dS = (\partial S/\partial K)\, dk + (\partial S/\partial a)da. \qquad \ldots(21)$$

The second term on the right-hand side signifies the change of S due to the change of a alone and therefore has to be identified with the negative of dE, i.e.,

$$dE = - (\partial S/\partial a)da. \qquad \ldots(22)$$

We thus find, using eqns. (20), (21) and (22),

$$W = - (\partial S/\partial K)dK = - (\partial S/\partial K) \sum_{n=1}^{2} (\partial K/\partial u^n)du^n \qquad \ldots(23)$$

and comparison of this result with eqn. (19) gives

$$U_n = - (\partial S/\partial K)\,(\partial K/\partial u^n), \quad n = 1,2. \qquad \ldots(24)$$

The equations of motion can thus be written as

$$d(\partial T/\partial \dot{u}^n)/dt - \partial T/\partial u^n = - (\partial S/\partial K)\,(\partial K/\partial u^n), \quad n = 1,2, \qquad \ldots(25)$$

or, if we use quasi-Cartesian coordinates

$$m\ddot{x}^n \approx - (\partial S/\partial K)\,(\partial K/\partial x^n), \quad n = 1,2. \qquad \ldots(25')$$

If a is a given function of t, $a = a(t)$, eqn. (25) is a system of second-order partial differential equations for the unknowns $u^1(t)$, $u^2(t)$.

A different and simpler problem is to assume that the curve traced out by the body on the surface is known, say $u^1 = f(u)^2$ is given, and that $a = a(u^2)$ has to be found. It can be shown that $a(u^2)$ exists for any curve which is not normal to the curves defined by $K =$ const.

The generalized force U_n can be derived from a potential iff

$$\begin{aligned} 0 &= \partial U_1/\partial u^2 - \partial U_2/\partial u^1 \\ &= (\partial^2 S/\partial K\partial a)\,[(\partial K/\partial u^2)\,(\partial a/\partial u^1) - (\partial K/\partial u^1)\,(\partial a/\partial u^2)]. \end{aligned} \qquad \ldots(26)$$

This is the case if

$$\partial a/\partial u^1 = \partial a/\partial u^2 = 0, \qquad \ldots(27)$$

i.e., if a is kept constant, or if

$$\partial K/\partial u^1 = \partial K/\partial u^2 = 0, \qquad ...(28)$$

i.e., if the curvature of the surface is constant and therefore the force $U_n = 0$.

6. Comparison with General Relativity

Let us now look at the similarities and differences between the motion of a body in G.R. and the motion of a two-dimensional prestressed body—such as the previous example – in a surface according to N.M.

A. *Similarities*

(1) The space (i.e., surface in the case of the the previous example, space-time in the case of G.R.) is curved and only the intrinsic geometry is relevant.

(2) Rigid motion of a body, i.e., a motion in which the distances between all particles of body remain unchanged, is generally not possible.

(3) A total force acting on a body cannot be well defined, since we do not have unique parallel propagation of vectors in a curved space.

(4) The mathematical complexities are in both cases overwhelming unless numerous approximations are introduced. Exact treatment of the motion of an elastic membrane on a surface would involve boundary value problems with moving boundary.

B. *Differences and why they are (mostly) ignorable*

(1) In G.R. the geometry of the space is influenced by the bodies present, whereas the space of the prestressed body is assumed to be given. *However*: It is in G.R. frequently assumed that the influence of a sufficiently light body on the geometry is negligible; such a body is called a test body. (The above applies to Mercury's motion around the sun.)

(2) Some books on G.R. state that test particles follow geodesics (principle of equivalence) whereas a prestressed body can change its motion by changing its self-stress. *However* : Proofs of the geodesic hypothesis have to assume that stresses in the test body are negligible and that the test body is small (Synge 1964). This assumption in our example corresponds to ignorable size and self-stress of the body, in which case the body does trace out a geodesic on the surface (Synge and Schild 1964)

(3) The space of G.R. is four dimensional, has an indefinite metric and physical time is measured by proper time, whereas the space for our example of a prestressed body is two dimensional with positive-definite metric and physical time is measured by Newton's absolute time. *However*: From experience with special relativity (S.R.) we may expect that in a static space-time and for slow motion the difference between absolute time (coordinate time) and proper time is small. The difference in dimension will produce significant effects. In our example we only needed the Gaussian curvature; in the case of G.R. for a static space-time we will have to work with all six independent components of the Riemann tensor of the hypersurface t = constant.

7. Hybrid Theories of N.M. and G.R.

Physical theories are models for some aspects of reality and their domain of validity is found by experiments. G.R. seems to be valid for arbitrary velocities and matter distributions, but is not valid in small domains. S.R. is a special case of G.R. obtained by assuming that space-time is flat, therefore considering all forms of matter as test bodies, but allowing arbitrary velocities. N.M. might be obtained from S.R. by assuming small velocities as well ($v \ll c \Rightarrow$ coordinate time of S.R. $\rightarrow$ absolute time of N.M.) with gravitation added by the introduction of the Newtonian potential (and its gradient). The results of sections 12-5 indicate that, besides S.R. and N.M., another special case of G.R. deserves consideration: G.R. specialized to slow motion of a prestressed test body in a static space-time or slow motion of a prestressed test body in a curved three-space with absolute time. Such a theory, restricted to the case where self-straining is large (and not controlled) has been considered by Pechlaner 1973. Many other thecries of gravitation located somewhere between N.M. and G.R. can be found in the literature (see Yilmaz 1975, Bsitp 1964).

8. Conjectures

The fact that mass densities have not even entered our calculations implies that the effect of self-straining on motion is not related to a tidal force. The effect of mass on motion is almost always dominant; otherwise it would be possible to pull oneself up by one's own bootstraps. Pulling on one's bootstraps is, of course, an example of self-straining.

Some situations where the influence of self-straining might show up in a body's motion, or in the force necessary to prevent motion, are now considered:

(1) Stresses in elementary particles are of the same order as mass-densities. If the elementary particle itself is large, such as a charged particle together with its Coulomb field, the effect of self-straining might become noticeable. Experiments on the fall of electrons in a gravitational field have been made (Witteborn and Fairbank 1967) but are difficult to interpret. Theories (Witteborn and Fairbank 1967, Pechlaner 1975) find it difficult to separate electromagnetic effects from gravitational effects.

(2) The motion of the moon can be predicted with large accuracy; but calculations are of a semi-empirical nature. Including effects of the moon's self-stress might improve our understanding of the physical librations of the moon (Marsden and Cameron 1966) and other phenomena.

(3) Motions of and on the earth might be affected by self-stress. Prediction and well as understanding of the Chandler wobble (Marsden and Cameron 1966), continent drift (Wegener 1924) and earthquakes might improve if self-stress is considered.

(4) Dicke's test of the principle of equivalence (Misner *et al.* 1973) repeated with bodies having large selfstress might show a violation of the principle of equivalence.

(5) Stationary satellites presently require rockets to keep them in location. Large overrigid space-trusses in which self-stress such as thermal stress is generated judiciously might make such rockets unnecessary.

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