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AXISYMMETRIC DISTRIBUTION OF THERMAL STRESSES IN A TRANSVERSELY ISOTROPIC SEMI-INFINITE SOLID CONTAINING A PENNY-SHAPED CRACK

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This paper contains an analysis of stresses and displacements in a transversely isotropic semi-infinite solid containing a penny-shaped crack, situated parallel to the free boundary. It is assumed that the boundary is at zero temperature, free from the surface traction and the crack is opened by the application of temperature to its surfaces. The problem has been reduced to solving a Fredholm integral equation of the second kind and a set of two simultaneous Fredholm integral equations of the second kind.

1. INTRODUCTION

In recent years, owing to the increased usage of anisotropic materials in engineering applications, the interest in anisotropic elasticity has grown up considerably. But a few mixed boundary-value problems of the steady state thermal stresses of anisotropic materials have been studied yet.

A general method of solution of the steady state thermal stress problem of a transversely isotropic semi-infinite solid was given by Sharma (1958) first. Following Sharma's approach, a few problems by Tauchert and Aköz (1972, 1974) and by Murata and Atsumi (1977) have been discussed recently. In this paper, using such method as Sharma's, stresses in a transversely isotropic semi-infinite elastic medium containing a penny-shaped crack, situated parallel to the free boundary at a finite distance when the free edge is at zero temperature, free from surface traction with a temperature prescribed on the crack surface, are discussed. The plane of the crack thus divides the semi-infinite space into two domains, on one side of it is a semi-infinite space and on the other side is a slab of finite thickness. As in similar problems, it has been assumed that the temperature, displacements and stresses vanish at infinity and the thermal conditions on the upper surface of the crack are identical with those on the lower surface.

Numerical calculations for a crack opened by constant temperature have been carried out for some practical materials, β-quartz, cadmium, Magnesium crystals. The effect of transverse isotropy on stress-intensity factors is shown by comparing these results with those of Palaiya's (1969) isotropic case.

2. Temperature Field

The position of a typical point is expressed in terms of cylindrical coordinates (r, θ, z) taking centre of the crack to be the origin of the coordinates and the z-axis parallel to the material axis of symmetry. Let unity be the dimensionless radius of the crack.

Now the plane of the crack divides the solid into two domains, namely,

(i) the layer $-h \leqslant z \leqslant 0$ and

(ii) the half-space $0 \leqslant z < \infty$ (Fig. 1)

where h is the dimensionless thickness of the layer.

The steady state temperature field without thermal sources is determined by the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z'^2}\right) T(r, z) = 0 \qquad \ldots(2.1)$$

where $z' = z/k$ and $k^2(=k_z/k_r)$ is the ratio of coefficients of conductivity along z-axis and in z-plane.

The appropriate solution of (2.1) for semi-infinite solid $(0 \leqslant z < \infty)$ may be taken as

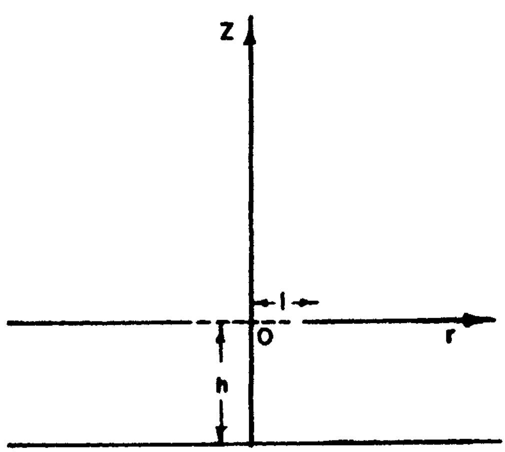

Fig. 1.

$$T(r, z) = \int_0^\infty \xi A e^{-\xi z'} J_0(\xi r)\, d\xi \qquad \text{...(2.2)}$$

and that for the layer $-h \leqslant z \leqslant 0$ may be written as

$$T(r, z) = \int_0^\infty \xi B \sinh \xi(z' + h') J_0(\xi r)\, d\xi \qquad \text{...(2.3)}$$

where $h' = h/k$ and A, B are functions of ξ be determined from the thermal boundary conditions.

The thermal boundary conditions may be stated as follows:

$$T(r, -h) = 0, \quad \forall\, r \qquad \text{...(2.4)}$$

$$T(r, 0+) = T(r, 0-) = -T_0, \quad 0 \leqslant r < 1. \qquad \text{...(2.5)}$$

In addition to the equations (2.4), (2.5) the conditions of continuity of temperature and heat flux outside the crack on $z = 0$ may be written as

$$T(r, 0+) = T(r, 0-), \left.\frac{\partial T}{\partial z}\right|_{z=0+} = \left.\frac{\partial T}{\partial z}\right|_{z=0-}, \quad 1 < r < \infty. \qquad \text{...(2.6)}$$

Utilising the above boundary conditions, we see that

$$A = B \sinh \xi h' \qquad \text{...(2.7)}$$

and the dual integral equations for determining the unknown function A are

$$\int_0^\infty \xi^{-1}\alpha(1 - e^{-2\xi h'}) J_0(\xi r)\, d\xi = -2T_0, \quad 0 \leqslant r < 1 \qquad \text{...(2.8)}$$

$$\int_0^\infty \alpha J_0(\xi r)\, d\xi = 0, \quad r > 1 \qquad \text{...(2.9)}$$

where

$$\alpha = \xi^2(1 + \coth \xi h')\, A$$

The solution of the above dual integral equations as given by Sneddon (1966) is

$$\alpha = \xi \int_0^1 \beta(t) \cos \xi t\, dt \qquad \text{...(2.10)}$$

where $\beta(t)$ is determined from the Fredholm integral equation:

$$\beta'(t) - \int_0^1 \beta'(r) R'(r, t)\, dr = -\frac{4}{\pi} \qquad \text{...(2.11)}$$

where

$$\beta'(t) = \beta(t)/T_0, \; R'(r, t) = \frac{2}{\pi}\int_0^\infty e^{-2\xi h'} \cos \xi r \cos \xi t \, d\xi$$

$$= \frac{2h'}{\pi}\left[\frac{1}{4h'^2 + (r+t)^2} + \frac{1}{4h'^2 + (r-t)^2}\right]. \quad \ldots(2.12)$$

3. Thermal Stresses

Since the problem is axi-symmetric one, we have

$$u_r = u_r(r, z), \quad u_\theta = 0, \quad u_z = u_z(r, z) \quad \ldots(3.1)$$

where u_r, u_θ, u_z are the components of the displacement in the r, θ, z directions respectively.

The stress components are related to the displacements as follows

$$\left.\begin{aligned} \sigma_{rr} &= c_{11}\frac{\partial u_r}{\partial r} + c_{12}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} - b_1 T \\ \sigma_{\theta\theta} &= c_{11}\frac{u_r}{r} + c_{12}\frac{\partial u_r}{\partial r} + c_{13}\frac{\partial u_z}{\partial z} - b_1 T \\ \sigma_{zz} &= c_{13}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + c_{33}\frac{\partial u_z}{\partial z} - b_2 T \\ \tau_{rz} &= c_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \end{aligned}\right\} \quad \ldots(3.2)$$

where

$$b_1(c_{11} + c_{12})\,\alpha_1 + c_{13}\alpha_2, \; b_2 = 2c_{13}\alpha_1 + c_{33}\alpha_2 \quad \ldots(3.3)$$

and c_{ij} are elastic constants, T is the temperature of the solid from the state of zero stress and strain; α_1, α_2 are the coefficients of linear expansion in z-plane and along z-axis.

The displacement equations of equilibrium are

$$c_{11}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) + (c_{13} + c_{44})\frac{\partial^2 u_z}{\partial r \partial z} + c_{44}\frac{\partial^2 u_r}{\partial z^2} = b_1\frac{\partial T}{\partial r}$$

$$c_{44}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + (c_{13} + c_{44})\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) + c_{33}\frac{\partial^2 u_z}{\partial z^2} = b_2\frac{\partial T}{\partial z}. \quad \ldots(3.4)$$

To solve (3.6), we introduce the displacement potentials $\varphi(r, z)$ and $\psi(r, z)$ as in Sharma (1958).

Hence (3.4) gives

$$c_{11}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+\{c_{44}+\lambda(c_{13}+c_{44})\}\frac{\partial^2\varphi}{\partial z^2}=0 \qquad ...(3.5)$$

$$(\lambda c_{44}+c_{13}+c_{44})\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+\lambda c_{33}\frac{\partial^2\varphi}{\partial z^2}=0 \qquad ...(3.6)$$

$$c_{11}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\{c_{44}+\mu(c_{13}+c_{44})\}\frac{\partial^2\psi}{\partial z^2}=b_1T \qquad ...(3.7)$$

$$(\mu c_{44}+c_{13}+c_{44})\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\mu c_{33}\frac{\partial^2\psi}{\partial z^2}=b_2T \qquad ...(3.8)$$

where λ, μ are constants.

To solve the equations (3.7) and (3.8), we assume a solution of the form

$$\psi=\int_0^\infty \xi A\theta_1(\xi)\,e^{-\xi z'}J_0(\xi r)\,d\xi \qquad ...(3.9)$$

for the semi-infinite solid and that for the layer $-h \leqslant z \leqslant 0$ as

$$\psi=\int_0^\infty \xi B\theta_2(\xi)\sinh\xi(z'+h')\,J_0(\xi r)\,d\xi. \qquad ...(3.10)$$

It can be verified by direct substitution that this function ψ satisfies eqns. (3.7) and (3.8) providing

$$\mu=\frac{b_2(c_{44}-c_{11}k^2)+b_1(c_{13}+c_{44})\,k^2}{b_1(c_{33}-c_{44}k^2)-b_2(c_{13}+c_{44})} \qquad ...(3.11)$$

and

$$\xi^2\theta_1(\xi)=\xi^2\theta_2(\xi)=\frac{k^2\{b_1(c_{33}-c_{44}k^2)-b_2(c_{13}+c_{44})\}}{k^2(c_{13}+c_{44})^2+(c_{44}-k^2c_{11})(c_{33}-c_{44}k^2)}=P \text{ (say)} \qquad ...(3.12)$$

where P is a constant.

For getting a non-zero solution of (3.5) and (3.6), introducing displacement potentials $\varphi_1(r, z)$ and $\varphi_2(r, z)$ as in Sharma (1958), the displacement and stress components may be put as

$$u_r=\frac{\partial}{\partial r}(\varphi_1+\varphi_2+\psi),\quad u_z=\frac{\partial}{\partial z}(\lambda_1\varphi_1+\lambda_2\varphi_2+\mu\psi) \qquad ...(3.13)$$

(a) $$\sigma_{rr}=\left(c_{11}\frac{\partial^2}{\partial r^2}+c_{12}\frac{1}{r}\frac{\partial}{\partial r}\right)(\varphi_1+\varphi_2+\psi)+c_{13}\frac{\partial^2}{\partial z^2}(\lambda_1\varphi_1+\lambda_2\varphi_2+\mu\psi)-b_1T$$

(*equation continued on p.* 81)

(b) $$\sigma_{\theta\theta} = \left(c_{12}\frac{\partial^2}{\partial r^2} + c_{11}\frac{1}{r}\frac{\partial}{\partial r}\right)(\varphi_1 + \varphi_2 + \psi) + c_{13}\frac{\partial^2}{\partial z^2}(\lambda_1\varphi_1 + \lambda_2\varphi_2 + \mu\psi) - b_1 T$$

(c) $$\sigma_{zz} = c_{13}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\varphi_1 + \varphi_2 + \psi) + c_{33}\frac{\partial^2}{\partial z^2}(\lambda_1\varphi_1 + \lambda_2\varphi_2 + \mu\psi) - b_2 T$$

(d) $$\tau_{rz} = c_{44}\frac{\partial^2}{\partial r \partial z}[(1 + \lambda_1)\varphi_1 + (1 + \lambda_2)\varphi_2 + (1 + \mu)\psi] \quad \text{...(3.14)}$$

where λ_1 and λ_2 are values of λ corresponding to ν_1^2 and ν_2^2 which are the roots of the quadratic equation

$$c_{11}c_{44}\nu^4 + (2c_{13}c_{44} - c_{11}c_{33} + c_{13}^2)\nu^2 + c_{33}c_{44} = 0 \quad \text{...(3.15)}$$

and φ_1 and φ_2 are solutions of

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_i^2}\right)\varphi_i = 0 \quad (i = 1, 2) \quad \text{...(3.16)}$$

with $z_i = z/\nu_i$.

A general solution of (3.16) suitable for the semi-infinite solid is

$$\varphi_i = \int_0^\infty \xi^{-1} S_i(\xi) e^{-\xi z_i} J_0(\xi r)\, d\xi \quad \text{...(3.17)}$$

and that for the layer $-h \leqslant z \leqslant 0$ is

$$\varphi_i = \int_0^\infty \xi^{-1}\{T_i(\xi)\sinh \xi(z_i + h_i) + Q_i(\xi)\cosh \xi(z_i + h_i)\} J_0(\xi r)\, d\xi \quad \text{...(3.18)}$$

where

$$h_i = h/\nu_i$$

The boundary conditions may be stated as follows:

$$\left.\begin{aligned} &\sigma_{zz}(r, -h) = \tau_{rz}(r, -h) = 0, \quad \forall r \\ &\sigma_{zz}(r, 0+) = \sigma_{zz}(r, 0-) = 0, \quad 0 \leqslant r < 1 \\ &\tau_{rz}(r, 0+) = \tau_{rz}(r, 0-) = 0, \quad 0 \leqslant r < 1 \end{aligned}\right\} \quad \text{...(3.19)}$$

In addition to these, for $z = 0$ to pass through the region unoccupied by the crack, the values of displacements, stresses should all be continuous. This requires the following additional boundary conditions:

$$u_r(r, 0+) = u_r(r, 0-),\ u_z(r, 0+) = u_z(r, 0-), \quad 1 < r < \infty$$

$$\sigma_{zz}(r, 0+) = \sigma_{zz}(r, 0-),\ \tau_{rz}(r, 0+) = \tau_{rz}(r, 0-), \quad 1 < r < \infty. \qquad \ldots(3.20)$$

Utilising the boundary conditions (3.19), (3.20) with some minor manipulations, we get the following system of simultaneous integral equations for determining the unknowns M and N:

$$\int_0^\infty M J_1(\xi r)\, d\xi = 0, \quad r > 1 \qquad \ldots(3.21)$$

$$\int_0^\infty N J_0(\xi r)\, d\xi = 0, \quad r > 1 \qquad \ldots(3.22)$$

$$\int_0^\infty \{(\nu_1 + \nu_2)(e^{-\xi h_1} - e^{-\xi h_2})^2 M - (\nu_1 + \nu_2)(\nu_1 e^{-2\xi h_1} + \nu_2 e^{-2\xi h_2}) N + 4\nu_1\nu_2 e^{-\xi(h_1+h_2)} N + (\nu_1 - \nu_2)^2 N\}\, \xi J_0(\xi r)\, d\xi = F(r), \quad 0 \leqslant r < 1 \qquad \ldots(3.23)$$

$$\int_0^\infty \left\{-\frac{\nu_1 + \nu_2}{\nu_1\nu_2}(\nu_1 e^{-2\xi h_2} + \nu_2 e^{-2\xi h_1}) M + 4e^{-\xi(h_1+h_2)} M + \frac{(\nu_1 - \nu_2)^2}{\nu_1\nu_2} M + (\nu_1 + \nu_2)(e^{-\xi h_1} - e^{-\xi h_2})^2 N\right\} \xi J_1(\xi r)\, d\xi = G(r), \quad 0 \leqslant r < 1 \qquad \ldots(3.24)$$

where

$$M = \left(\sinh \xi h_1 - \frac{\nu_2}{\nu_1}\frac{1 + \lambda_1}{1 + \lambda_2}\sinh \xi h_2\right) T_1(\xi) + \left(\cosh \xi h_1 - \frac{1 + \lambda_1}{1 + \lambda_2} \times \cosh \xi h_2\right) Q_1(\xi) + \left(\sinh \xi h' - \frac{\nu_2}{k}\frac{1 + \mu}{1 + \lambda_2}\sinh \xi h_2\right)\frac{PA}{\sinh \xi h'} - S_1(\xi) - S_2(\xi) - PA,$$

$$N = \left(\frac{\lambda_1}{\nu_1}\cosh \xi h_1 - \frac{\lambda_2}{\nu_1}\frac{1 + \lambda_1}{1 + \lambda_2}\cosh \xi h_2\right) T_1(\xi) + \left(\frac{\lambda_1}{\nu_1}\sinh \xi h_1 - \frac{\lambda_2}{\nu_2}\frac{1 + \lambda_1}{1 + \lambda_2}\sinh \xi h_2\right) Q_1(\xi) + \left(\frac{\mu}{k}\cosh \xi h' - \frac{\lambda_2}{k}\frac{1 + \mu}{1 + \lambda_2} \times \cosh \xi h_2\right)\frac{PA}{\sinh \xi h'} + \frac{\lambda_1}{\nu_1} S_1(\xi) + \frac{\lambda_2}{\nu_2} S_2(\xi) + \frac{\mu}{k} PA,$$

$$F(r) = -\frac{2\nu_1(\lambda_1 - \lambda_2)}{(1 + \lambda_1)(1 + \lambda_2)}\int_0^\infty \left[\left(1 - \frac{\nu_2}{\nu_1}\right)(1 + \mu) + \frac{\nu_2}{k}(1 + \mu) \times\right.$$

(*equation continued on p.* 83)

$$\times (e^{-\xi h_2} - e^{-\xi h_1}) \operatorname{cosech} \xi h' + \frac{1 + \coth \xi h'}{(\lambda_1 - \lambda_2)k} \Big\langle (\mu - \lambda_1)(1 + \lambda_2)$$

$$\times \left\{ -\nu_1 e^{-\xi(h_1+h_2)} + \frac{\nu_1 - \nu_2}{2\nu_1} \nu_2 + \frac{(\nu_1 + \nu_2)\nu_2}{2\nu_1} e^{-2\xi h_2} \right\} + (\mu - \lambda_2)(1 + \lambda_1)$$

$$\times \left\{ -\nu_2 e^{-\xi(h_1+h_2)} - \frac{\nu_1 - \nu_2}{2} + \frac{\nu_1 + \nu_2}{2} e^{-2\xi h_1} \right\} \Big\rangle \Big] PA\, \xi_0(J\xi r)\, d\xi \qquad \ldots(3.25)$$

and

$$G(r) = -\frac{2(\lambda_1 - \lambda_2)}{(1 + \lambda_1)(1 + \lambda_2)} \int_0^\infty \Big[\frac{1 + \mu}{k} (\nu_1 - \nu_2) \coth \xi h' + \frac{1 + \mu}{k}$$

$$\times (\nu_2 e^{-\xi h_1} - \nu_1 e^{-\xi h_2}) \operatorname{cosech} \xi h' + \frac{1 + \coth \xi h'}{(\lambda_1 - \lambda_2)\, k} \Big\langle (\mu - \lambda_1)(1 + \lambda_2)$$

$$\times \left\{ \nu_2 e^{-\xi(h_1+h_2)} + \frac{\nu_1 - \nu_2}{2} - \frac{\nu_1 + \nu_2}{2} \cdot e^{-2\xi h_2} \right\} + (\mu - \lambda_2)(1 + \lambda_1)$$

$$\times \left\{ \nu_1 e^{-\xi(h_1+h_2)} - \frac{\nu_1 - \nu_2}{2} - \frac{\nu_1 + \nu_2}{2} e^{-2\xi h_1} \right\} \Big\rangle \Big]$$

$$\times PA\, \xi J_1(\xi r)\, d\xi. \qquad \ldots(3.26)$$

Taking the trial solutions

$$M = \xi^{1/2} \int_0^1 m(t)\, J_{3/2}(t)\, dt = -\left(\frac{2}{\pi}\right)^{1/2} \int_0^1 t^{1/2} m(t) \frac{d}{dt}\left(\frac{\sin \xi t}{\xi t}\right) dt \qquad \ldots(3.27)$$

$$N = \int_0^1 n(t) \sin \xi t\, dt = -\frac{\cos \xi}{\xi} n(1) + \frac{1}{\xi} \int_0^1 \frac{dn(t)}{dt} \cos \xi t\, dt \qquad \ldots(3.28)$$

we see that eqns. (3.21) and (3.22) are satisfied automatically and eqns. (3.23), (3.24) after some manipulations utilising a few integral formulae (Erdelyi 1954) yield

$$n'(t) - \frac{2}{\pi} \int_0^1 \{m'(r)\, R(t, r) + n'(r)\, S(t, r)\}\, dr = -\frac{4}{\pi} \int_0^1 \beta'(r)\, Q_1'(t, r)\, dr \qquad \ldots(3.29)$$

$$m'(t) - \int_0^1 \{m'(r)\, R_1(t, r) + n'(r)\, S_1(t, r)\}\, dr = -2 \int_0^1 \beta'(r)\, Q_2'(t, r)\, dr \qquad \ldots(3.30)$$

where

$$m'(t) = \frac{(\nu_1 - \nu_2)^2(1+\lambda_1)(1+\lambda_2)}{P\nu_1\nu_2(\lambda_1 - \lambda_2)} \frac{m(t)}{T_0}, \quad n'(t)$$
$$= \frac{(\nu_1 - \nu_2)^2(1+\lambda_1)(1+\lambda_2)}{P\nu_1\nu_2(\lambda_1 - \lambda_2)} \frac{n(t)}{T_0},$$

$$R(t, r) = -\frac{\nu_1+\nu_2}{(\nu_1-\nu_2)^2}\left(\frac{2}{\pi r}\right)^{1/2}\Big[\frac{1}{r}\{G_2(2h_1, t, r) + G_2(2h_2, t, r)$$
$$-2G_2(h_1 + n_2, t, r)\} - G_3(2h_1, r, t) - G_3(2h_2, r, t)$$
$$+2G_3(h_1 + h_2, r, t)\Big],$$

$$S(t, r) = \frac{\nu_1+\nu_2}{(\nu_1-\nu_2)^2}[\nu_1 G_4(2h_1, r, t) + \nu_2 G_4(2h_2, r, t)$$
$$- 4\nu_1\nu_2 G_4(h_1 + h_2, r, t)],$$

$$R_1(t, r) = \frac{2\nu_1\nu_2}{\pi(\nu_1-\nu_2)^2}\left(\frac{t}{r}\right)^{1/2}\Big[-\frac{2(\nu_1+\nu_2)}{\nu_1\nu_2 rt}\{h_2\nu_1 G_2(2h_2, t, r)$$
$$+ h_1\nu_2 G_2(2h_1, t, r)\} + \frac{4}{rt}(h_1 + h_2)\, G_2(h_1 + h_2, t, r)$$
$$+ \frac{\nu_1+\nu_2}{\nu_1\nu_2}\{\nu_1 G_5(2h_2, t, r) + \nu_2 G_5(2h_1, t, r)\}$$
$$-4G_5(h_1 + h_2, t, r)\Big],$$

$$S_1(t, r) = -\left(\frac{2t}{\pi}\right)^{1/2}\frac{\nu_1\nu_2(\nu_1+\nu_2)}{(\nu_1-\nu_2)^2}\Big[\frac{1}{t}\{G_2(2h_1, t, r) + G_2(2h_2, t, r)$$
$$-2G_2(h_1 + h_2, t, r)\} - \{G_3(2h_1, t, r) + G_3(2h_2, t, r)$$
$$-2G_3(h_1 + h_2, t, r)\}\Big],$$

$$Q_1'(t, r) = \tfrac{1}{2}\frac{\nu_1-\nu_2}{\nu_1\nu_2}\Big[(1+\mu) + \frac{1}{(\lambda_1-\lambda_2)k}\{(\mu-\lambda_1)(1+\lambda_2)\nu_2$$
$$- (\mu - \lambda_2)(1+\lambda_1)\,\nu_1\}\Big]$$
$$\times\left|\begin{matrix} \pi/2 & r < t \\ \pi/4 & r = t \\ 0 & r > t \end{matrix}\right| - \tfrac{1}{2}\frac{\nu_1-\nu_2}{\nu_1\nu_2}(1+\mu)\, G_1(2h', t, r)$$
$$+ \frac{1+\mu}{k}\{G_1(h_2 + h', t, r) - G_1(h_1 + h', t, r)\} -$$

(equations continued on p. 85)

$$- \frac{1}{(\lambda_1 - \lambda_2)k} \{(\mu - \lambda_1)(1 + \lambda_2) + (\mu - \lambda_2)(1 + \lambda_1)\}$$

$$\times G_1(h_1 + h_2, t, r) + \frac{1}{(\lambda_1 - \lambda_2)k} \frac{\nu_1 + \nu_2}{2\nu_1\nu_2}$$

$$\times \{\nu_2(\mu - \lambda_1)(1 + \lambda_2) G_1(2h_2, t, r) + \nu_1(\mu - \lambda_2)(1 + \lambda_1)$$

$$\times G_1(2h_1, t, r)\},$$

$$Q_2'(t, r) = -\left(\frac{2}{\pi t}\right)^{1/2} \Big[\frac{1 + \mu}{2k} (\nu_1 - \nu_2)\{2h' G_1(2h', t, r) + rG_2(2h', t, r)\}$$

$$+ \frac{1 + \mu}{2k} \nu_2 \{(h_1 + h') G_1(h_1 + h', t, r) + rG_2(h_2 + h', t, r)\}$$

$$- \frac{1 + \mu}{k} \nu_1 \{(h_2 + h') G_1(h_2 + h', t, r) + rG_2(h_2 + h', t, r)\}$$

$$+ \frac{1}{(\lambda_1 - \lambda_2)k} \{(\mu - \lambda_1)(1 + \lambda_2)\nu_2 + (1 + \lambda_1)(\mu - \lambda_2)\nu_1\}$$

$$\times \{(h_1 + h_2) G_1(h_1 + h_2, t, r) + rG_2(h_1 + h_2, t, r)\}$$

$$- \frac{\nu_1 + \nu_2}{2(\lambda_1 - \lambda_2)k} (\mu - \lambda_1)(1 + \lambda_2) \{2h_2 G_1(2h_2, t, r)$$

$$+ rG_2(2h_2, t, r)\} - \frac{\nu_1 + \nu_2}{2(\lambda_1 - \lambda_2)k} (\mu - \lambda_2)(1 + \lambda_1)$$

$$\times \{2h_1 G_1(2h_1, t, r) + rG_2(2h_1, t, r)\}\Big],$$

$$G_1(x, y, z) = \tfrac{1}{2} \tan^{-1} \frac{2xy}{x^2 - y^2 + z^2}, \quad G_2(x, y, z) = \tfrac{1}{4} \log \frac{x^2 + (y + z)^2}{x^2 + (y - z)^2},$$

$$G_3(x, y, z) = \frac{z(x^2 + y^2 + z^2)}{\{x^2 + (y + z)^2\}\{x^2 + (y - z)^2\}},$$

$$G_4(x, y, z) = \frac{2xyz}{\{x^2 + (y + z)^2\}\{x^2 + (y - z)^2\}},$$

$$G_5(x, y, z) = \frac{x(x^2 + y^2 + z^2)}{\{x^2 + (y + z)^2\}\{x^2 + (y - z)^2\}}.$$

Once the solutions of the integral equations (3.29) and (3.30) are obtained, the stress intensity factors can be calculated as follows:

$$\rho = \lim_{r \to 1+} (r - 1)^{1/2} [\sigma_{zz}(r, 0)]_{r>1} = -\frac{\nu_1\nu_2}{\nu_1 - \nu_2} \frac{c_{44} T_0 P}{2\sqrt{2}} n'(1) \qquad \ldots(3.31)$$

$$\sum = \lim_{r \to 1+} (r - 1)^{1/2} [\tau_{rz}(r, 0)]_{r>1} = \frac{C_{44} T_0 P}{2(\nu_1 - \nu_2)\sqrt{\pi}} m'(1) \qquad \ldots(3.32)$$

Also we can easily see that

ρ_∞ (i.e. the normal stress intensity factor as $h \to \infty$) =

$$-\frac{\sqrt{2}}{\pi}\left[(1+\mu) + \frac{1}{(\lambda_1 - \lambda_2)k}\{(\mu - \lambda_1)(1+\lambda_2)\,\nu_2 - (\mu - \lambda_2) \times (1+\lambda_1)\,\nu_1\}\right] T_0 c_{44} P. \qquad \ldots(3.33)$$

4. Numerical Analysis

Numerical calculations were carried out on a high speed computer, IBM 370/155, in the Computer Centre of I. I. T., Madras.

The values of the elastic constants c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, the coefficients of linear expansion, α_1, α_2, the ratio k^2 of coefficients of conductivity for Cadmium, Beta-quartz, Magnesium used for numerical calculations are given in Table I. The integral equation (2.11) was solved by Fox and Goodwin (1953) method by dividing the interval (0, 1)

Table I

Values of physical constants (the elastic constants c_{ij} are in units of 10^{10} N/m² and the coefficients of linear expansion α_i in units of 10^{-5}/°C)

	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	k^2	α_1	α_2
Cadmium	11.0	4.04	3.83	4.69	1.56	1.0	54.0	20.2
β-quartz	11.66	1.67	3.28	11.04	3.61	0.51	7.0	12.9
Magnesium	5.97	2.62	2.17	6.17	1.64	1.0	27.7	26.6

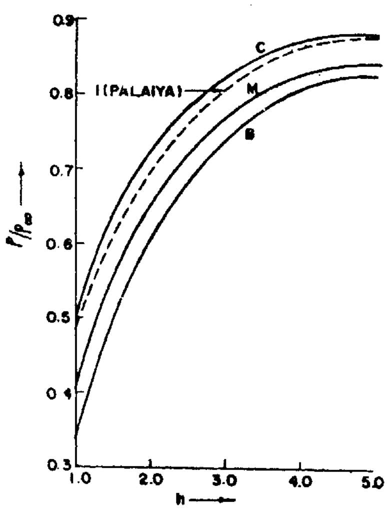

Fig. 2.

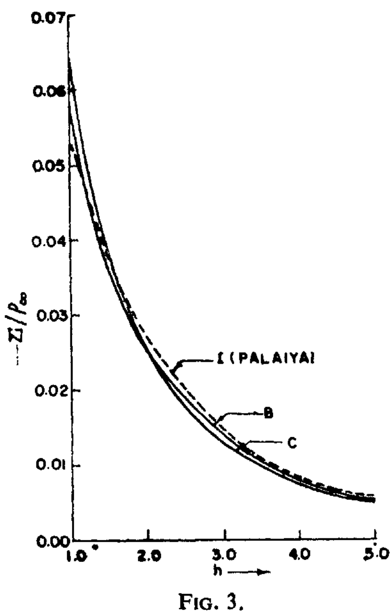

Fig. 3.

into 10 equal parts with pivotal points at 0, 0.1, 0.2, ..., 0.9, 1.0 for different values of h. For each value of h, we got the values of $\beta'(t)$ at pivotal points by solving a system of eleven linear algebraic simultaneous equations. Following the same method each of the integral equations (3.29) and (3.30) was placed by a system of ten simultaneous linear equations since $m'(0) = n'(0) = 0$. Solving this system of twenty simultaneous linear equations for every values of h, we got value of $m'(t)$ and $n'(t)$ at the pivotal points.

The ratios of the stress intensity factors i.e. ρ/ρ_∞ and $-\Sigma/\rho_\infty$ were calculated for $h = 1.05, 1.10, 1.20, 1.30, 1.6667, 2.5, 5.0$.

The numerical results are presented in Figs. 2 and 3. In the graphs *I, C, B, M* stand for isotropic, cadmium, beta-quartz and magnesium respectively. It is seen that for each material ρ/ρ_∞ increases as h increases and $-\Sigma/\rho_\infty$ decreases as h increases. The ratio, ρ/ρ_∞ of normal stress intensity factors is ordered in the following manner : cadmium > isotropic > magnesium > beta-quartz. While analysing the case of $-\Sigma/\rho_\infty$, we saw that the cases of Beta-quartz and Magnesium were very close to each other differing at the fourth place after the decimal point. That is why these two cases are shown by the same graph.

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