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SOME PERFECT FLUID SPHERES WITH NON-ZERO SPIN DENSITY

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In this paper some exact interior solutions describing perfect fluid distribution in the frame work of Einstein-Cartan theory are presented. A static spherically symmetric line element is considered. Unlike the usual assumptions in general relativity here the pressure is discontinuous on the boundary of the fluid sphere. The constants are evaluated by matching the solutions at the boundary to the Schwarzschild exterior metric.

INTRODUCTION

In recent years an extension of the theory of general relativity has been studied by Trautman (1972 a, b) and Hehl (1973, 1974a, b) following the suggestion by Cartan (1922) that the basic geometry of the space-time may be non-Riemannian with non-symmetric affinities (or the torsions) with the intrinsic spin density of matter. Thus while the field equations differ from those of general relativity in the presence of spinning particles, the torsions vanish in the vacuum and the field equations also retain their old form. Consequently, Birkhoff's theorem, that in case of spherical symmetry the metric in empty space may be reduced to the Schwarzschild form, remains valid in the new formalism, commonly called the Einstein-Cartan theory. Exact cosmological solutions have been derived (Hehl 1973, 1974a, b; Cartan 1922; Kuchowicz 1950) which demonstrate the possibility of avoiding the 'big bang' singularity (Trautman 1973). These models have little or no shear and no magnetic fields. Hehl *et al.* (1976) pointed out that these assumptions are unrealistic, particularly at high magnetic fields. The problem of gravitational collapse of stars in an empty space back-ground on the other hand does not seem to have received adequate attention so far.

Prasanna (1975) found the analogue of some static spherically symmetric solutions of Tolman (1939) in Einstein-Cartan theory. In this solution the equations corresponding to the classical condition of hydrostatic equilibrium was rather artificially split up into two separate equations so that a so-called 'spin conservation equation' is satisfied. In effect this meant that spin does not contribute any force (in the classical sense) to the equilibrium. But this assumption led to the difficulty that the derivatives of the components of the metric tensor could not be made continuous on the boundary, which besides being a violations of the Lichnerowicz (1975), seems difficult to accept. The junction conditions were later discussed at some length by Kuchowicz (1975b) and Arkuszewski *et al.* (1975). According to the junction conditions of the last mentioned workers the pressure on the surface of a Weyssenhoff fluid (1947) sphere does not necessarily vanish. They however reached the erroneous conclusion that such a sphere does not bounce. The error was pointed out by Hehl *et al.* (1976).

Adopting Hehl's approach (1973, 1974a) to Einstein-Cartan theory, Prasanna (1975) has obtained the solution analogues to these obtained by Tolman (1939) in general relativity for static fluid spheres. He has found that a space-time metric similar to the Schwarzschild interior solution will no longer represent a homogeneous fluid sphere in the presence of spin density.

In this paper we have solved the Einstein-Cartan equations for a static spherically symmetric fluid sphere by a suitable assumption on the metric potential g_{11}. We have taken $e^{\lambda} = Ar^n$ and have obtained some solutions for general value of n.

2. The Field Equations

The Einstein-Cartan equations for the ideal fluid are

$$G_{ij} = -8\pi T_{ij} \qquad \text{...(2.1a)}$$

where

$$T^i_j = [(\rho + p)\, u^i u_j - g^i_j\, p]_{u^i u_j = 1}. \qquad \text{...(2.1b)}$$

The only non-vanishing components of T^i_j are $T^4_4 = \rho$, $T^1_1 = T^2_2 = T^3_3 = -p$, we take the spherically symmetric line element

$$ds^2 = e^{\nu(r)}\, dt^2 - e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad \text{...(2.2)}$$

Here G_{ij} is the Einstein tensor and T_{ij} is the energy momentum tensor. p and ρ represent pressure and density respectively.

It then follows from eqns. (2.1a) and (2.1b) that the field equations are [Prasanna (1975), Hehl (1973, 1974)]

$$8\pi\rho = 16\pi^2 K^2 + \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) \qquad \text{...(2.3)}$$

$$8\pi p = 16\pi^2 K^2 - \frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) \qquad \text{...(2.4)}$$

$$8\pi p = 16\pi^2 K^2 + e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right) \qquad \text{...(2.5)}$$

where

$$K = A_1 e^{-\nu/2}. \qquad \text{...(2.6)}$$

K is called the spin density of the distribution and A_1 is constant of integration and a prime denotes differentiation with respect to r.

It is clear from these equations that it is the $\bar{p}$ and not the p which is continuous across the boundary $r = R_b$ of the fluid sphere. The continuity of $\bar{p}$ across the boundary ensures that of $\nu' \exp(\nu)$. Further, with $\bar{p}$ and $\bar{\rho}$ replacing p and ρ respectively, we are assured that the metric coefficients are continuous across the boundary. Hence we shall apply the usual boundary conditions to the solutions of eqns. (2.3), (2.4) and (2.5).

The exterior metric is taken as usual Schwarzschild line element given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \qquad \text{...(2.7)}$$

where M is the total mass of the fluid sphere.

The total mass, as measured by an external observer, inside the fluid sphere of radiaus R_b is given by

$$M = 4\pi \int_0^{R_b} \bar{\rho} \, r^2 \, dr.$$

$$M = 4\pi \int_0^{R_b} \rho r^2 \, dr - 8\pi^2 \int_0^{R_b} K^2(r) \, r^2 dr. \qquad \text{...(2.8)}$$

Thus the total mass of fluid sphere is modified by the correction

$$8\pi^2 \int_0^{R_b} K^2(r) \, r^2 dr.$$

We define

$$\nu = 2 \log y \qquad \text{...(2.9)}$$

Using eqns. (2.4) and (2.5) we get the differential equation

$$y'' - \left(\frac{1}{r} + \frac{\lambda'}{2}\right) y' + \left(\frac{e^{\lambda}}{r^2} - \frac{\lambda'}{2r} - \frac{1}{r^2}\right) y = 0. \qquad \text{...(2.10)}$$

It is not always possible to get a traceable solution from the analytic specification of the equation of state. In these cases numerical and graphic techniques are easy to apply. Exact solutions in terms of known functions are most easily obtained by requiring one of the field variables to satisfy some subsidiary conditions which simplify the full set of equations Adler (1974) and Whiteman (1977).

Once the field equations are solved, an equation of state then can be extracted. Such solutions may be useful in understanding a system in the extreme relativistic limit where we cannot specify a priori what the equation of state might be.

3. Solutions of the Field Equations

Equations (2.10) cannot be solved without either choosing an equation of state or making a specific assumption on one of the function p, ρ, λ and ν. For this we assume

$$e^{\lambda(r)} = Ar^n \qquad \text{...(3.1)}$$

where A and n are constants. Substitution of eqn. (3.1) in (2.10) provides

$$y'' - \left(\frac{1}{r} + \frac{n}{2r}\right) y' + \left(Ar^{n-2} - \frac{n}{2r^2} - \frac{1}{r^2}\right) y = 0. \qquad \text{...(3.2)}$$

This is a second order differential equation in y for the general value of n and A. We solve it in the following cases:

Case 1 : *When* $n = 0$

In this case eqn. (3.2) assumes the form

$$r^2 y'' - ry' + (A - 1)\, y = 0. \qquad \text{...(3.3)}$$

On putting $P = -1$ and $\theta = A - 1$ (3.3) is transformed into

$$r^2 y'' + Pry' + \theta y = 0 \qquad \text{...(3.4)}$$

which can be recognized as Euler's equation.

The solution of eqn. (3.3) may now be written down and the metric potential $\nu(r)$ is obtained.

To solve eqn. (3.3) or (3.4) there are the following possible cases (Ritzeran and Rose 1968).

Case 1*a* : $2 > A > 1$

The solutions are

$$y = B_1 r^l + C_1 r^m, \quad e^\lambda = A \qquad \text{...(3.5)}$$

where B_1 and C_1 are constants to be fixed by the boundary conditions and

$$l = 1 + \eta, m = 1 - \eta \quad \text{for } \eta = \sqrt{2 - A}. \qquad \text{...(3.6)}$$

Pressure and density are given by

$$8\pi pr^2 = 16\pi^2 \; A_1^2 r^2 (B_1 r^{1+\eta} + C_1 r^{1-\eta})^{-2} - 1$$
$$+ A^{-1} [B_1(3 + 2\eta)\, r^{2n} + C_1(3 - 2\eta)]\, (B_1 r^{2\eta} + C_1)^{-1} \qquad \text{...(3·7)}$$

$$8\pi\rho r^2 = 1 - A^{-1} + 16\pi^2 A_1^2\, r^2 (B_1 r^{1+\eta} + C_1 r^{1-\eta})^{-2}. \qquad \text{...(3.8)}$$

The spin density K is given by

$$K = A_1 e^{-\nu/2} = A_1 (B_1 r^{1+\eta} + C_1 r^{1-\eta})^{-1}. \qquad \text{...(3.9)}$$

The constants A_1, B_1, C_1 appearing in the solution can be obtained by matching the solution at the boundary ($r = R_b$)

$$A^{-1} = \left(1 - \frac{2M}{R_b}\right). \qquad \text{...(3.10)}$$

[from continuity of g_{rr} using (2.2), (2.7) and (3.1)]

$$(B_1 R_b^l + C_1 R_b^m)^2 = (1 - 2MR_b^{-1}) \qquad \text{...(3.11)}$$

[from continuity of g_{tt} using (2.2) (2.7) and (3.5)]

and

$$2(B_1 R_b^l + C_1 R_b^m)(B_1 l R_b^{l-1} + C_1 m R_b^{m-1}) = 2MR_b^{-2} \qquad \text{...(3.12)}$$

[from continuity of $\frac{\partial}{\partial r}(g_{tt})$ using (2.2), (2.7) and (3.5)].

Equations (3.10), (3.11) and (3.12) may be solved for A, B_1, C_1 yielding

$$A = (1 - 2MR_b^{-1})^{-1} \qquad \text{...(3.13)}$$

$$B_1 = \frac{M}{2\eta R_b^{l+1}\left(1 - 2MR_b^{-1}\right)^{1/2}}\left[1 - R_b m M^{-1}\left(1 - 2MR_b^{-1}\right)\right] \qquad \text{...(3.14)}$$

$$C_1 = -\frac{M}{2\eta R_b^{m+1}\left(1 - 2MR_b^{-1}\right)^{1/2}}\left[1 - R_b l M^{-1}\left(1 - 2MR_b^{-1}\right)\right]. \qquad \text{...(3.15)}$$

Also A_1 is determined from

$$8\pi\rho(R_b)\, R_b^2 = 16\pi^2 A_1^2\, R_b^2\, (B_1 R_b^{1+\eta} + C_1 R_b^{1-\eta})^{-2} + 1 - A^{-1}. \qquad \text{...(3.16)}$$

Case 1*b* : $A = 2$

The solution in this case is

$$y = r(B_2 \log r + C_2) \qquad \text{...(3.17)}$$

where B_2 and C_2 are constants.

Pressure and density are

$$8\pi p r^2 = 16\pi^2 A_1^2 (B_2 \log r + C_2)^{-2} - 1 + \tfrac{1}{2}\{B_2(3 \log r + 2) + 3C_2\}(B_2 \log r + C_2)^{-1} \qquad \text{...(3.18)}$$

$$8\pi\rho r^2 = \tfrac{1}{2} + 16\pi^2 A_1^2 (B_2 \log r + C_2)^{-2}. \qquad \ldots(3.19)$$

The spin density K is

$$K = A_1 e^{-\nu/2} = A_1 R_b^{-1} (B_2 \log R_b + C_2)^{-1}. \qquad \ldots(3.20)$$

The constants B_2 and C_2 are given by

$$B_2 = \frac{1}{R_b\left(1 - 2MR_b^{-1}\right)^{1/2}} \left[3MR_b^{-1} - 1\right] \qquad \ldots(3.21)$$

$$C_2 = \frac{\left(1 - 2MR_b^{-1}\right)^{1/2}}{R_b} - \frac{\left(3MR_b^{-1} - 1\right) \log R_b}{R_b\left(1 - 2MR_b^{-1}\right)^{1/2}}. \qquad \ldots(3.22)$$

Also A_1 is given by

$$8\pi\rho(R_b)\, R_b^2 = \tfrac{1}{2} + 16\pi^2 A_1^2 (B_2 \log R_b + C_2)^{-2}. \qquad \ldots(3.23)$$

Case 1*c* : *When* $A > 2$

$$y = r^{\sigma} [B_3 \cos x + C_3 \sin x], \; e^{\lambda} = A \qquad \ldots(3.24)$$

where $x = \delta \log r$ and σ, δ will depend on A.

From eqns. (2.3) and (2.4) pressure and density are given by

$$8\pi p r^2 = 16\pi^2 A_1^2 r^{2-2\sigma} (B_3 \cos x + C_3 \sin x)^{-2} + \frac{1}{A}\left[1 + 2\left\{\frac{(\sigma B_3 + \delta C_3)\cos x + (\sigma C_3 - \delta B_3)\sin x}{B_3 \cos x + C_3 \sin x}\right\}\right] - 1 \qquad \ldots(3.25)$$

$$8\pi\rho r^2 = 16\pi^2 A_1^2 r^{2-2\sigma} (B_3 \cos x + C_3 \sin x)^{-2} + 1 - A^{-1}. \qquad \ldots(3.26)$$

The spin density K is

$$K = A_1 e^{-\nu/2} = \frac{A_1 (B_3 \cos x + C_3 \sin x)^{-1}}{r^{\sigma}}. \qquad \ldots(3.27)$$

The constants B_3 and C_3 are given by

$$B_3 = \cos x_0 R_b^{-\sigma}\left(1 - \frac{2M}{R_b}\right)^{1/2} - \frac{\sin x_0}{2R_b^{\sigma+1}\left(1 - 2MR_b^{-1}\right)^{1/2}} \times \left[1 - 2(1+\sigma)\left(1 - 2MR_b^{-1}\right)\right] \qquad \ldots(3.28)$$

$$C_3 = \sin x_0 R_b^{-\sigma}\left(1 - 2MR_b^{-1}\right)^{1/2} + \frac{\cos x_0}{2R_b^{\sigma+1}\left(1 - 2MR_b^{-1}\right)^{1/2}}$$

$$\times\left[1 - 2(1+\sigma)\left(1 - 2MR_b^{-1}\right)\right] \qquad \text{...(3.29)}$$

where $\qquad x_0 = \delta \log R_b.$

Also A_1 is given by

$$8\pi\rho(R_b)R_b^2 = 16\pi^2 A_1^2 \; R_b^{2-2\sigma} (B_3 \cos x + C_3 \sin x)^{-2} + 1 - \frac{1}{A}. \qquad \text{...(3.30)}$$

Case 2 : When $n = -2$

Equation (3.2) reduces to

$$y'' + \frac{A}{r^4} y = 0. \qquad \text{...(3.31)}$$

which has the solution

$$y = (2/\pi\sqrt{A})^{1/2} \; r \cos(\sqrt{A}/r). \qquad \text{...(3.32)}$$

Thus

$$e^{\nu} = (2/\pi\sqrt{A}) \; r^2 \cos^2 (\sqrt{A}/r)$$

$$e^{\lambda} = A/r^2.$$

Pressure and density are given by

$$8\pi p r^2 = 4\pi^3 A_1^2 \sqrt{A} \cdot \frac{1}{\cos^2(\sqrt{A}/r)^{-1}} + \frac{3r^2}{A} + \frac{2r}{\sqrt{A}} \tan(\sqrt{A}/r). \qquad \text{...(3.33)}$$

$$8\pi\rho r^2 = 8\pi^3 A_1^2 \sqrt{A} \frac{1}{\cos^2(\sqrt{A}/r)} + 1 - \frac{3r^2}{A}. \qquad \text{...(3.34)}$$

Also K is given by

$$K = A_1 e^{-\nu/2} = A_1 (2/\pi\sqrt{A})^{-1/2} r^{-1} \cos^{-1}(\sqrt{A}/r). \qquad \text{...(3.35)}$$

The constant A is given by

$$A = \left(1 - 2MR_b^{-1}\right)^{-1} R_b^2 . \qquad \text{...(3.36)}$$

A_1 is given by

$$8\pi\rho(R_b)R_b^2 = \frac{8\pi^3 A_1^2 \sqrt{A}}{\cos^2(\sqrt{A}/R_b)} + 1 - \frac{3R_b^2}{A}. \qquad \text{...(3.37)}$$

In addition the conditions $p \geqslant 0$ and $\rho \geqslant$ will impose further restrictions on our solutions. We therefore restrict our solutions to only those values of constants for which the pressure and density are positive.

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