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EXISTENCE AND UNIQUENESS THEOREMS FOR FOUR POINT BOUNDARY VALUE PROBLEMS FOR FOURTH ORDER DIFFERENTIAL EQUATIONS

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In this paper existence and uniqueness theorems for certain four point boundary value problems for fourth order differential equations are proved.

1. INTRODUCTION

This paper gives criteria for the existence and uniqueness of solutions to four-point boundary value problems associated with the differential equation

$$y^{(iv)} = f(x, y, y', y'', y''') \qquad \ldots(1.1)$$

where $f(x, y, z, v, w)$ is assumed to be continuous on a subset of R^5, assuming that the solutions of the initial value problems associated with (1.1) exist, are unique and can be extended throughout a fixed subinterval of R. In section 2, a monotonicity condition on f ensures that the following two-point boundary value problems

$$y^{(iv)} = f(x, y, y', y'', y''')$$

$$y(x_1) = y_1, y(x_2) = y_2\ y^{(i)}(x_2) = m,\ y'''(x_2) = k\ (i = 1, 2); \qquad \ldots(1.2_i)$$

$$[y^{(1)}(x_2) = y'(x_2),\ y^{(2)}(x_2) = y''(x_2)]$$

$$y^{(iv)} = f(x, y, y', y'', y''') \qquad \ldots(1\ 3_i)$$

$$y(x_2) = y_2\ y^{(i)}(x_2) = m,\ y'''(x_2) = k,\ y(x_3) = y_3\ (i = 1, 2);$$

$$y^{(iv)} = f(x, y, y', y'', y''),$$

$$y(x_2) = y_2\ y^{(i)}(x_3) = p,\ y'''(x_2) = k,\ y(x_3) = y_3\ \ (i = 1, 2) \qquad \ldots(1.4_i)$$

and

$$y^{(iv)} = f(x, y, y', y'', y'''), \qquad \ldots(1.5_i)$$

$$y(x_3) = y_3,\ y^{(i)}(x_3) = p,\ y'''(x_3) = k,\ y(x_4) = y_4\ ;$$

have atmost one solution, and with an additional hypothesis a unique solution to the four-point boundary value problem is constructed. This is accomplished by matching the solutions of the boundary value problems (1.2_2), (1.3_2), (1.4_2) and (1.5_2). Section 2 also gives sufficient conditions under which solutions exist to (1.2_i), (1.3_i), $(1\ 4_i)$ and (1.5_i) $(i = 1, 2)$. Sufficient conditions are investigated for the existence and uniqueness for solutions to three-point boundary value problems by using the matching technique developed in Rao and Murty (1981).

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS TO FOUR-POINT BOUNDARY VALUE PROBLEMS

This section gives a criterion under which solutions of (1.1) which satisfy the boundary conditions at two points may be matched to obtain a unique solution of four-point boundary value problems. It will be assumed throughout that f is a continuous function on $[x_1, x_4] \times R^4$, where $[x_1, x_4]$ is a closed subinterval of R. The next theorem establishes the idea of matching solutions of four two-point boundary value problems which yield a unique solution of a certain class of four point boundary value problems.

Theorem 2.1—Let $y_1, y_2, y_3, y_4, m, k, p, k_1 \in R$ with $x_1 < x_2 < x_3 < x_4$ and suppose that

(1) there exist solutions of (1.2_i), (1.3_i), (1.4_i) and (1.5_i) $(i = 1, 2)$

(2) for each $m \in R$ there exists atmost one solution for each of the following two-point boundary value problems

$$y^{(iv)} = f(x, y, y', y'', y''') \qquad \ldots(2.1)$$

$$y(x_1) = y_1, y(x_2) = y_2\ y''(t) = m, y''(x_2) = k\ t \in (x_1, x_2];$$

$$y^{(iv)} = f(x, y, y', y'', y''') \qquad \ldots(2.2)$$

$$y(x_2), = y_2, y''(t) = m, y'''(x_2) = k, y(x_3) = y_3$$

when $t \in [x_2, x_3]$; and

$$y^{(iv)} = f(x, y, y', y'', y''') \qquad \cdots(2.3)$$

$$y(x_3) = y_3, y''(t) = p. y'''(x_2) = k, y(x_4) = y_4, t \in [x_3, x_4).$$

Then there exists a unique solution of the four point boundary value problem

$$y^{(iv)} = f(x, y, y', y'', y''') \qquad \ldots(2.4)$$

$$y(x_1) = y_1, y(x_2) = y_2, y(x_3) = y_3, y(x_4) = y_4.$$

PROOF : Let $\phi(x,m)$ denote a solution of (1.2_2) with second derivative m at $x = x_2$. It will be first shown that $\phi'(x_2,m)$ is an increasing continuous function of m. By definition, if $m_1 < m_2$ then

$$\phi''(x_2, m_1) < \phi''(x_2, m_2).$$

Moreover, it is plain that

$$\phi''(x, m_1) < \phi''(x, m_2)$$

for all $x \in (x_1, x_2]$. On the contrary, suppose that there exists a point $r \in (x_1, x_2)$ such that $\phi''(r, m_1) \geqslant \phi''(r, m_2)$. Since $\phi''(x, m_2)$ and $\phi''(x, m_1)$ are continuous functions there exists a point $r_1 \in [r, x_2]$ such that $\phi''(r_1, m_1) = \phi''(r_1, m_2)$. This is a contradiction to the hypothesis 2. Since $\phi(x_1, m_1) = \phi(x_1, m_2)$ and $\phi(x_2, m_1) = \phi(x_2, m_2)$ there exists an $r_2 \in (x_1, x_2)$, such that $\phi'(r_2, m_1) = \phi'(r_2, m_2)$. Moreover, this together with $\phi'(x_2, m_1) = \phi'(x_2, m_2)$ gives that there exists an $r_3 \in [r_2, x_2]$ such that $\phi''(r_3, m_1) = \phi''(r_3, m_2)$. This together with the above claim implies that $\phi''(x, m_2) > \phi''(x, m_1)$ for all $x \in (r_3, x_2]$. In particular $\phi''(x_2, m_2) > \phi''(x_2, m_1)$.

Hence $\phi'(x_2, m)$ when considered as a function of m is strictly increasing. Let $\Psi(x, m)$ denote the solution of (1.3_2) with third derivative m at $x = x_2$. The trend of reasoning given above demonstrates that $\Psi''(x_2, m)$ when considered as a function of m is strictly decreasing. Since solutions exist to (1.2_i) and (1.3_i) $(i = 1, 2)$, it follows that neither $\phi''(x_2, m)$ nor $\Psi^n(x_2, m)$ has jump discontinuities as functions of m and thus both $\phi''(x_2, m)$ and $\Psi''(x_2, m)$ are strictly continuous monotonic functions of m whose range is the set of reals. Hence there exists a unique $m_0 \in R$ such that $\phi''(x_2, m_0) = \Psi''(x_2, m_0)$.

Let $\theta(x, p)$ denote the solution of (1.4_2) with second derivative p at $x = x_3$. Then it can be shown similarly that $\theta''(x_3, p)$ when considered as a function of p is a strictly increasing continuous function of p whose range is the set of reals. Let $\eta(x, p)$ denote a solution of (1.5_2) with second derivative p at $x = x_3$. It can be shown that $\eta''(x_3, p)$ is a strictly decreasing continuous function of p whose range is the set of reals. Hence there exists a unique p_0 such that

$$\theta''(x_3, p_0) = \eta''(x_3, p_0).$$

From the solutions of (1.3_2) and (1.4_2) we have $\Psi(x_2, m_0) = \theta(x_2, p_0)$, $\Psi'''(x_2, m_0) = \theta'''(x_2, p_0)$, $\Psi(x_3, m_0) = \theta(x_3, m_0)$ and $\Psi''(x_2, m_0) = m_0$, $\theta''(x_2, p_0) = p_0$. If $m_0 \neq p_0$, then θ and Ψ are distinct solutions of (2.2), which is a contradiction to hypothesis 2 in Theorem 2.1. Therefore $\Psi(x) = \theta(x) \ \forall \ x \in [x_2, x_3]$. Hence $\chi(x)$ defined by

$$\chi(x) = \begin{cases} \phi(x, m_0), & x_1 \leqslant x \leqslant x_2 \\ \Psi(x, m_0) = \theta(x, p_0), & x_2 \leqslant x \leqslant x_3 \\ \eta(x, p_0), & x_3 \leqslant x \leqslant x_4 \end{cases}$$

is a solution of (2.4).

Now to establish uniqueness, suppose ϕ and Ψ are distinct solutions of (2.4). Write $\chi(x) = \phi(x) - \Psi(x)$. Then clearly, $\chi(x_1) = \chi(x_2) = \chi(x_3) = \chi(x_4) = 0$. This implies that there exists an $\alpha \in (x_1, x_2)$, $\beta \in (x_2, x_3)$ and $\gamma \in (x_3, x_4)$ such that $\chi'(\alpha) = 0$, $\chi'(\beta) = 0$ and $\chi'(\gamma) = 0$. These properties again imply that there exists an $\alpha_1 \in (\alpha, \beta)$ and $\alpha_2 \in (\beta, \gamma)$ such that $\chi''(\alpha_1) = 0$ and $x''(\alpha_2) = 0$ which again imply that there exists an $\alpha_3 \in (\alpha_2, \alpha_3)$ such that $\chi'''(\alpha_3) = 0$ which is a contradiction to the hypothesis 2. Thus uniqueness is established.

The next theorem establishes the validity of hypothesis 1.

Theorem 2.2—Let $f : [x_1, x_4] \times R^4 \to R$ and suppose that there exists an $N > 0$ such that $|f(x, y, y', y'', y''')| \leqslant N$. Then for any $y_1, y_2, y_3, y_4, m, m_1, p \in R$, there exist solutions of (1.2_i), (1.3_i), (1.4_i) and (1.5_i) for $(i = 1,2)$.

The proof of this theorem is straightforward and hence omitted (Bailey *et al.* 1966, Barr and Sherman 1973).

Theorem 2.3—Let f be continuous and satisfy a Lipshitz condition with positive constants k_0, k_1, k_2 and k_3 and

$$\alpha = \frac{1}{24} K_0 h^4 + \frac{\sqrt{3}}{27} k_1 h^3 + \frac{1}{6} k_2 h^2 + \frac{2}{3} k_3 h < 1 \qquad \ldots(2.5)$$

where $h = x_2 - x_1$. Then given $x_1, x_2, y_1, y_2, y_3, m, K \in R$ with $x_1 \leqslant x \leqslant x_2$ there exists a unique solution of the boundary value problem (1.2_i) $(i = 1.2_i)$.

PROOF : The proof of the boundary value problem (1.2_2) will be given. The proof of (1.2_1) is analogous. Consider the boundary value problem

$$y^{(iv)} = 0 \qquad \ldots(2.6)$$

$$y(x_1) = 0,\ y(x_2) = 0,\ y''(x_2) = 0,\ y'''(x_2) = 0.$$

This problem has no non-zero solution. Therefore, if $h(x)$ is any continuous function on $[x_1, x_2]$, then a solution of

$$y^{(iv)} = h(x)$$

satisfying the boundary conditions (2.6) is given by

$$y(x) = \int_{x_1}^{x_2} G(x, s)\, h(s)\, ds,$$

where $G(x, s)$ is the Green's function of the boundary value problem. It can easily be shown that

$$\max_{x_1 \leqslant x \leqslant x_2} \int_{x_1}^{x_2} |\, G(x,s)\,|\, ds \leqslant \frac{(x_2 - x_1)^4}{12}$$

$$\max_{x_1 \leqslant x \leqslant x_2} \int_{x_1}^{x_2} |\, G_x(x,s)\,|\, ds \leqslant \frac{\sqrt{3}}{27}\,(x_2 - x_1)^3$$

$$\max_{x_1 \leqslant x \leqslant x_2} \int_{x_1}^{x_2} |\, G_{xx}(x,s)\,|\, ds \leqslant \frac{1}{6}\,(x_2 - x_1)^2$$

$$\max_{x_1 \leqslant x \leqslant x_2} \int_{x_1}^{x_2} |\, G_{xxx}(x,s)\,|\, ds \leqslant \frac{2}{3}\,(x_2 - x_1).$$

Let B be the complete normed linear space of thrice continuously differentiable functions on $[x_1, x_2]$ with norm

$$\|y\| = \max_{x_1 \leqslant x \leqslant x_2} [k_0\,|\,y(x)\,| + k_1\,|\,y'(x)\,| + k_2\,|\,y''(x)\,| + k_3\,|\,y'''(x)\,|],$$

Define the operator T by

$$T(y(x)) = \int_a^b G(x,s)\, f(s, y(s), y'(s), y''(s))\, ds.$$

Using the above bounds on $G(x, s)$, it can be shown that

$$\|\, T(y_1) - T(y_2)\,\| \leqslant \alpha\, \|\, y_1 - y_2 \,\|$$

and if $\alpha < 1$, then T is a contraction operator, and by the Banach fixed point theorem T has a unique solution of (1.1) satisfying the boundary conditions in (2.6). Now to find a unique solution of (1.2_2), let $p(x)$ be a polynomial of degree three such that $p(x_1) = y_1, p(x_2) = y_2, p''(x_2) = m, p'''(x_2) = k$. By adopting the above procedure to the boundary value problem

$$y^{(iv)} = f(x, y + p(x), (y + p(x))', (y + p(x))'', (y + p(x))'''),$$

$$y(x_1) = 0, y(x_2) = 0, y''(x_2) = m, y'''(x_2) = k,$$

a unique solution $y_1(x)$ is obtained. Define $y(x) = y_1(x) + p(x)$. Then we get a unique solution of (1.2_2).

Analogous results follow to the other boundary value problems, (1.3_i), (1.4_i) and (1.5_i) ($i = 1, 2$).

Theorem 2.4 — Let f be continuous and satisfy a Lipshitz condition on $[x_1, x_3] \times R^4$, and suppose $h_1 = x_2 - x_1$ and $h_2 = x_3 - x_2$ satisfy the inequality (1.9). Then given $y_1, y_2, m, y_3 \in R$ there exists a unique solution of the three-point boundary value problem

$$y^{(iv)} = f(x, y, y', y'', y''')$$

$$y(x_1) = y_1, y(x_2) = y_2, y'''(x_2) = m, y(x_3) = y_3.$$

PROOF: The proof utilises the matching technique developed by Rao and Murty (1981).

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