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THE SPECIAL PROJECTIVE TENSOR FIELDS (WITH 2-RECURRENT AND SPECIAL QUASI SYMMETRIC PROPERTIES) IN NON SYMMETRIC FINSLER SPACE F_n^*

S. B. MISRA AND ANAND KUMAR

Department of Mathematics, M.L.K. (P.G.) College, Balrampur 271201

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In an n-dimensional Finsler space the special projective tensor field has been defined by Kumar (1975) as

$$M_j^i(x, \dot{x}) = F(x, \dot{x})\ H_j^i + P(x, \dot{x})\ Q_j^i$$

which is a linear combination of projective entity $Q_j^i(x, \dot{x})$ and Berwalds's deviation tensor field $H_j^i(x, \dot{x})$, where P is homogeneous scalar function of degree one in $\dot{x}$. The tensor fields

$$M_{hj}^i = \frac{2}{3}\dot{\partial}_{[h}\ M_{j]}^i$$

and

$$M_{lhj}^i = \frac{2}{3}\dot{\partial}^2{}_{l[h} M_{j]}^i$$

have also been defined by Kumar. In the present communication we study the two recurrent and special quasi symmetric properties of these tensor fields in F_n^*.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n^* (Pandey and Gupta 1979) equipped with non-symmetric connection $\Gamma_{jk}^i(x, \dot{x}) \neq \Gamma_{kj}^i(x, \dot{x})$ based on non-symmetric fundamental tensor $g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x})$. In this paper we assume that non-symmetric connection $\Gamma_{jk}^i(x, \dot{x})$ is homogeneous of degree zero in its directional arguments $\dot{x}^l$'s.

The curvature tensor H_{hjk}^i arising from Berwald's covariant derivative as given in Rund (1959) is

$$H_{hjk}^i \neq \frac{2}{3}\ \dot{\partial}^2_{h[j} H_{k]}^i = 2\ \left\{\ \partial_{[k} G_{j]h}^i + G_{h[j}^r\, G_{k]r}^i + G_{rh[k}^i\ G_{j]}^r \right\}. \qquad \ldots(1.1)$$

*$2A_{[ij]} = A_{ij} - A_{ji}$, $2A_{(ij)} = A_{ij} + A_{ji}$

The projective entity Q^i_{hjk} arising from the projective covariant derivative as given in Misra (1966) is

$$Q^i_{hjk} = 2\left\{\dot{\partial}_{[k}\,\pi^i_{j]h} - \pi^i_{rh[j}\;\pi^r_{k]} + \pi^r_{h[j}\;\pi^i_{k]r}\right\} \qquad \ldots(1.2)$$

here

$$\pi^i_{hk}(x, \dot{x}) = G^i_{hk} - \frac{1}{n+1}\left(2\delta^i_{(h}\,G^r_{k)r} + \dot{x}^i\,G^r_{rkh}\right)$$

are called projective connection coefficients, positively homogeneous of degree zero in $\dot{x}^i$ and G^i, the Berwald's connection coefficient is positively homogeneous of degree two in $\dot{x}^i$.

The tensor fields H^i_{jkh} and H^i_{hi} satisfy the identities and contractions as has been given in Rund (1959, p. 129).

The projective entity Q^i_{jhk} satisfies the following relations

$$\text{(a) } Q^i_k\,\dot{x}^k = 0 \quad \text{(b) } \dot{\partial}_h\,Q^i_j\,\dot{x}^j = -\,Q^i_h \quad \text{(c) } Q^i_{jkh} = -\,Q^i_{jhk} \text{ and}$$

$$\text{(d) } Q^i_k = Q^i_{jk}\,\dot{x}^k = Q^i_{jhk}\,\dot{x}^h\,\dot{x}^k. \qquad \ldots(1.3)$$

The covariant derivative of an arbitrary tensor field T^i_j has been defined in Pandey and Gupta (1979) as

$$T^{i+}_{j+}\Big|_k = \partial_k T^i_j - \left(\dot{\partial}_m T^i_j\right)\Gamma^m_{pk}\,\dot{x}^p + T^m_j\,\Gamma^i_{mk} - T^i_m\,\Gamma^m_{jk}. \qquad \ldots(1.4)$$

The commutation formula involving the covariant derivative (1.4) is given by

$$T^{i+}_{j+}\Big|_{hk} - T^{i+}_{j+}\Big|_{kh} = -\left(\dot{\partial}_m T^i_j\right)R^m_{hk} + T^m_j\,R^i_{mhk} - T^i_m\,R^m_{jhk}$$

$$+\left(T^{i+}_{j+}\Big|_m\right)N^m_{kh} \qquad \ldots(1.5)$$

here $N^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$, and

$$R^i_{jkh} = \partial_l\Gamma^i_{jk} - \partial_k\,\Gamma^i_{jl} + \left(\dot{\partial}_m\,\Gamma^i_{jl}\right)\Gamma^m_{pk}\,\dot{x}^p - \left(\dot{\partial}_m\,\Gamma^i_{jk}\right)\Gamma^m_{pl}\,\dot{x}^p$$

$$+\,\Gamma^p_{jk}\,\Gamma^i_{pl} - \Gamma^p_{jl}\,\Gamma^i_{pk}, \qquad \ldots(1.6)$$

is curvature tensor and satisfies the following identities and contractions:

$$\text{(a) } R^i_l = \dot{x}^k\,R^i_{kl} = \dot{x}^k\,\dot{x}^j\,R^i_{jkl}, \quad \text{(b) } R = \frac{1}{n-1}\,R^i_i,$$

$$\text{(c) } R_{jk} = R^i_{jki}, \quad \text{(d) } R^i_{hjk} = -\,R^i_{hkj}, \quad \text{(e) } R^i_{jk} = -\,R^i_{kj}. \qquad \ldots(1.7)$$

The commutation formula involving the partial derivative and covariant derivative (1.4) is given by

$$\left(\dot{\partial}_h T_j^i\right)_+^+\Big|_k - \dot{\partial}_h\left(T_{j+}^{i+}\Big|_k \right) = \left(\dot{\partial}_m T_j^i\right)\left(\dot{\partial}_h \Gamma_{pk}^m\right) \dot{x}^p + T_m^i \left(\dot{\partial}_h \Gamma_{jk}^m\right) - T_j^m \left(\dot{\partial}_h \Gamma_{mk}^i\right). \qquad ...(1.8)$$

2. Special Projective Tensor Fields

The special projective tensor field M_j^i has been defined in Kumar (1975) as

$$M_j^i = F(x, \dot{x}) \; H_j^i + P(x, \dot{x}) \; Q_j^i \qquad ...(2.1)$$

here P is a scaler homogeneous function of degree one in its directional arguments $\dot{x}^i$.

The special projective tensor fields M_{hj}^i and M_{lhj}^i are respectively given by

$$\text{(a)} \quad M_{hj}^i = \frac{2}{3}\dot{\partial}_{[h} \; M_{j]}^i,$$

$$\text{(b)} \quad M_{lkj}^i = \dot{\partial}_l \; M_{hj}^i = \frac{2}{3}\dot{\partial}_{l[h} \; M_{j]}^i. \qquad ...(2.2)$$

With the help of above equations the special projective tensor fields M_{hj}^i and M_{lhj}^i can be expressed in the following alternative forms as

$$\text{(a)} \quad M_{hj}^i = FH_{hj}^i + PQ_{hj}^i + \frac{2}{3}\left[\dot{\partial}_{[h} \; PQ_{j]}^i + \dot{\partial}_{[h} \; FH_{j]}^i\right]$$

$$\text{(b)} \quad M_{lhj}^i = FH_{lhj}^i + PQ_{lhj}^i + (\dot{\partial}_l F) H_{hj}^i + (\dot{\partial}_l P) Q_{hj}^i + \frac{2}{3}\left\{\dot{\partial}_{l[h}^2 \; PQ_{j]}^i + \dot{\partial}_{[h}P\dot{\partial}_{<l>}Q_{j]}^i + 2g_{l[h} H_{j]}^i + \dot{\partial}_{[h} F\dot{\partial}_{<l>}H_{j]}^i\right\} \qquad ...(2.3)$$

(Indices in the bracket $\langle \; \rangle$ are free from symmetric and skew symmetric parts.)

The special projective tensor fields satisfy the following identities and contractions :

$$\text{(a)} \quad M_j^i\dot{x}^j = 0, \quad \text{(b)} \; M_{hj}^i\dot{x}^h = \frac{4}{3}M_j^i, \quad \text{(c)} \quad M_{lhj}^i\dot{x}^l = 2M_{hj}^i,$$

$$\text{(d)} \quad M_{lhj}^i\dot{x}^l \, \dot{x}^h = \frac{8}{3}M_j^i, \; \text{(e)} \quad M_i^i = (n-1)\, FH + PQ_i^i,$$

$$\text{(f)} \quad M_{hi}^i = PQ_h + FH_h + \frac{1}{3}\left[(n-1)(\dot{\partial}_h F)\, H + (\dot{\partial}_h P)\; Q_i^i - \left(\dot{\partial}_i PQ_h^i + \dot{\partial}_i FH_h^i\right)\right]. \qquad ...(2.4)$$

3. Definitions

Definition 3.1—An n-dimensional Finsler space F_n^* is said to be 1-recurrent if the special projective tensor field M_{lhj}^i satisfies the relation

$$M^{i}_{lhj+}{}^{+}\Big|_{k} = \lambda_k \; M^{i}_{lhj} \qquad \text{...(3.1)}$$

where λ_k is a non-null recurrence vector field.

Transvecting the above equation by $\dot{x}^l$ and using (2.4c), we have

$$M^{i+}_{hj+}\Big|_{k} = \lambda_k \; M^{i}_{hj} \, . \qquad \text{...(3.2)}$$

Again transvecting (3.2) by $\dot{x}^h$ and using (2.4b), we have

$$M^{i+}_{j+}\Big|_{k} = \lambda_k \; M^{i}_{j} \, . \qquad \text{...(3.3)}$$

Thus we conclude that if M^{i}_{lhj} is 1-recurrent then so are M^{i}_{hj} and M^{i}_{j}.

Definition 3.2—An n-dimensional Finsler space F^{*}_{n} is said to be 2-recurrent if the special projective tensor field M^{i}_{lhj} satisfies the relation

$$M^{i+}_{lhj+}\Big|_{km} = \beta_{km} \; M^{i}_{lhj} \qquad \text{...(3.4)}$$

where β_{km} is recurrence tensor field of second order.

Transvecting (3.4) by $\dot{x}^l$ and noting (2.4c) we have

$$M^{i+}_{hj+}\Big|_{km} = \beta_{km} \; M^{i}_{hj} \, . \qquad \text{...(3.5)}$$

Again transvecting it by $\dot{x}^h$ and using (2.4b), we get

$$M^{i+}_{j+}\Big|_{km} = \beta_{km} \; M^{i}_{j} \, . \qquad \text{...(3.6)}$$

Hence the special projective deviation tensor fields M^{i}_{hj} and M^{i}_{j} are 2-reccurent if so is M^{i}_{lhj}.

Definition 3.3—In an F^{*}_{n}, if the special projective tensor field M^{i}_{lhj} satisfies

$$M^{i+}_{lhj+}\Big|_{km} = 0.$$

Then the F^{*}_{n} is said to be special quasi symmetric (Kumar 1979).

4. 2-Recurrent Tensor Field

Differentiating (3.1) in the sense of (1.4) with respect to x^m, we get

$$M^{i+}_{lhj+}\Big|_{km} = \lambda^{+}_{k+}\Big|_{m} \, M^{i}_{lhj} + \lambda_k \; M^{i+}_{lhj+}\Big|_{m} \, . \qquad \text{...(4.1)}$$

Using eqns. (3.1) and (3.4) in the eqns. (4.1), we get

$$\beta_{km} M^{i}_{lhj} = \lambda_{k+}^{+}\Big|_{m} M^{i}_{lhj} + \lambda_k \lambda_m \; M^{i}_{lhj} \, . \qquad \text{...(4.2)}$$

Since M^{i}_{lhj} is an arbitrary tensor field hence above equation directly gives

$$\beta_{km} = \lambda^{+}_{k+}\Big|_{m} + \lambda_k \, \lambda_m \, . \qquad \text{...(4.3)}$$

Thus we have:

Theorem 4.1—In an F_n^*, in order that the 2-recurrent special projective tensor field be 1-recurrent, eqn. (4.3) must necessarily hold.

Using the special quasi symmetric property and relation (3.1) in (4.1) we get

$$0 = (\lambda_{k+}^{+}|_m + \lambda_k \lambda_m) M_{lhj}^i . \quad \ldots(4.4)$$

But since the special projective tensor field is non-zero, therefore (4.4) can be replaced by

$$\lambda_{k+}^{+}\Big|_m + \lambda_k \lambda_m = 0 . \quad \ldots(4.5)$$

Commutating (4.5) with respect to indices k and m, we get

$$\lambda_{k+}^{+}\Big|_m = \lambda_{m+}^{+}|_k . \quad \ldots(4.6)$$

In this way we can say that the recurrence vector is gradient, then we have

Theorem 4.2—In an special quasi symmetric F_n^*, the non-zero 1-recurrence vector will be a gradient.

Interchanging the indices k and m in eqn. (3.4) and substracting the equation thus obtained from (3.4) and using the commutation formula (1.5), we get

$$-\left(\dot{\partial}_r M_{lhj}^j\right) R_{km}^r + M_{lhj}^r R_{rkm}^i - M_{rhj}^i R_{lkm}^r - M_{lrj}^i R_{hkm}^r - M_{lhr}^i R_{jkm}^r + \left(M_{lhj+}^{i+}\Big|_r\right) N_{mk}^r = (\beta_{km} - \beta_{mk}) \; M_{lhj}^i . \quad \ldots(4.7)$$

Theorem 4.3—In an F_n^*, the special projective 2-recurrence tensor field is non symmetric.

Theorem 4.4—In a special quasi symmetric flat F_n^*, the special projective 2-recurrence tensor field satisfies the relation

$$\left(M_{lhj}^{i+}{}_{-}\Big|_r\right) N_{mk}^r - \left(\dot{\partial}_r M_{lhj}^i\right) R_{km}^r = 0. \quad \ldots(4.8)$$

Commutating the indices k and m in (3.4) and applying the commutation formula (1.5), we get

$$\left(\beta_{mk} - \beta_{km}\right) M_j^i = -\left(\dot{\partial}_l M_j^i\right) R_{mk}^l + M_j^l R_{lmk}^i - M_l^i R_{jmk}^r + \left(M_j^i\Big|_{+l}^{+}\right) N_{km}^l \quad \ldots(4.9)$$

contracting the equation (4.9) with respeet to indices i and j and using the relation (2.4e) and rearranging the terms in the equation thus obtained, we get

$$\beta_{km} - \beta_{mk} = \dot{\partial}_l \log\left[(n-1)\, FH + PQ_i^i\right] R_{mk}^l - \log\left[(n-1)\, FH + PQ_i^i\right]_{+}^{+}\Big|_l N_{km}^l . \quad \ldots(4.10)$$

Thus we have:

Theorem 4.5—In an F_n^*, the special projective 2-recurrence tensor field satisfies the relation (4.10).

Theorem 4.6—In a special quasi symmetric F_n^*, the special projective 2-recurrent tensor field satisfies the relation

$$\dot{\partial}_l \log\left[(n-1)\,FH + PQ_i^i\right] R^l_{mk} = \log\left[(n-1)\,FH + PQ_i^i\right]\overset{+}{\underset{+}{|}}_l N^l_{km}\,.$$

Commutating (3.5) with respect to indices k and m and using the commutation formula (1.5) in the equation thus obtained, we get

$$(\beta_{km} - \beta_{mk})\, M^i_{hj} = -\left(\dot{\partial}_r M^i_{hj}\right) R^r_{km} + M^r_{hj} R^i_{rkm} - M^i_{rj} R^r_{hkm}$$

$$- M^i_{hr} R^r_{jkm} + \left(M^{i+}_{hj+}|_r\right) N^r_{mk}\,. \qquad \ldots(4.11)$$

Differentiating (4.11) in the sense of (1.4) with respect to x^n and taking help of the equation (4.11) itself, and(1.8), and rearranging the terms, we get

$$(\beta_{km} - \beta_{mk})\overset{+}{\underset{+}{|}}_n M^i_{hj} + \left[\left(\dot{\partial}_p M^i_{hj}\right)\left(\dot{\partial}_r\, \Gamma^p_{qm}\right)\dot{x}^q + M^i_{pj}\left(\dot{\partial}_r \Gamma^p_{hn}\right)\right.$$

$$\left. + M^i_{hp}\left(\dot{\partial}_r \Gamma^p_{jn}\right) - M^p_{hj}\left(\dot{\partial}_r \Gamma^i_{pn}\right)\right] R^r_{km} + \left(\dot{\partial}_r M^i_{hj}\right) R^{r+}_{km+}|_n - M^r_{hj} R^{i+}_{rkm+}|_n$$

$$+ M^i_{rj} R^{r+}_{nkm+}|_n + M^i_{hr} R^{r+}_{jkm+}|_n - M^{i+}_{hj+}|_r\left(N^{r+}_{mk+}|_n + R^r_{nkm}\right) = 0. \qquad \ldots(4.12)$$

Thus we have

Theorem 4.7—In an F_n^*, the special projective recurrence tensor field β_{km} satisfies (4.12).

Theorem 4.8—In an F_n^* the special projective recurrent tensor field satisfies the following relation :

$$\text{(a)}\quad (\beta_{km} - \beta_{mk})\overset{+}{\underset{+}{|}}_n M^i_{hj} + \left(\dot{\partial}_r M^i_{hj}\right) R^{r+}_{km+}|_n - M^r_{hj} R^{i+}_{rkm+}|_n - M^i_{rj} R^{r+}_{hkm+}|_n$$

$$+ M^i_{hr} R^{r+}_{jkm+}|_n - M^{i+}_{hj+}|_r\left(N^{r+}_{mk+}|_n + R^r_{hkm}\right) = 0. \qquad \ldots(4.13)$$

if F_n^* is affinely connected.

$$\text{(b)}\quad (\beta_{km} - \beta_{mk})\overset{+}{\underset{+}{|}}_n M^i_{hj} + \left[\left(\dot{\partial}_p M^i_{hj}\right)\left(\dot{\partial}_r\, \Gamma^p_{qn}\right)\dot{x}^q + M^i_{pj}\left(\dot{\partial}_r\, \Gamma^p_{hn}\right)\right.$$

$$\left. + M^i_{hp}\left(\dot{\partial}_r\, \Gamma^p_{jn}\right) - M^p_{hj}\left(\dot{\partial}_r\, \Gamma^i_{pn}\right)\right] R^r_{km} + \left(\dot{\partial}_r M^i_{hj}\right) R^{r+}_{km+}|_n$$

$$- M^{i+}_{hj+}|_r N^{r+}_{mk+}|_n = 0 \qquad \ldots(4.14)$$

if F_n^* is flat one.

$$\text{(c)}\quad (\beta_{km} - \beta_{mk})\overset{+}{\underset{+}{|}}_n M^i_{hj} + \left(\dot{\partial}_r M^r_{hj}\right) R^{r+}_{km+}|_n - M^{i+}_{hj+}|_r N^{r+}_{mk+}|_n = 0 \qquad \ldots(4.15)$$

if F_n^* is flat affinely connected.

(d) $$\left[\left(\dot{\partial}_p M_{hj}^i\right)\left(\dot{\partial}_r \Gamma_{qn}^p\right)\dot{x}^q + M_{pj}^i\left(\dot{\partial}_r \Gamma_{hn}^p\right) + M_{hp}^i\left(\dot{\partial}_r \Gamma_{jn}^p\right) - M_{hj}^p\left(\dot{\partial}_r \Gamma_{pn}^i\right)\right] R_{kn}^r + \left(\dot{\partial}_r M_{hj}^i\right) R_{km+}^{r+}\Big|_n - M_{hj}^r R_{rkm+}^{i+}\Big|_n + M_{rj}^i R_{hkm+}^{r+}\Big|_n + M_{hr}^i R_{jkm+}^{r+}\Big|_n - M_{hj+}^{i+}\Big|_r \left(N_{mk+}^{r+}\Big|_n + R_{nkm}^r\right) = 0 \quad \ldots(4.16)$$

if F_n^* is special quasi symmetric.

(e) $$\left[\left(\dot{\partial}_p M_{hj}^i\right)\left(\dot{\partial}_r \Gamma_{qn}^p\right)\dot{x}^q + M_{pj}^i\left(\dot{\partial}_r \Gamma_{hn}^p\right) + M_{hp}^i\left(\dot{\partial}_r \Gamma_{jn}^p\right) - M_{hj}^p\left(\dot{\partial}_r \Gamma_{pn}^i\right)\right] R_{km}^r + \left(\dot{\partial}_r M_{hj}^i\right) R_{lm+}^{r+}\Big|_n - M_{hj+}^{i+}\Big|_r N_{mk+}^{r+}\Big|_n = 0 \quad \ldots(4.17)$$

if the space is special quasi symmetric and flat.

(f) $$\left(\dot{\partial}_r M_{hj}^i\right) R_{km+}^{r+}\Big|_n - M_{hj}^r R_{rkm+}^{i+}\Big|_n + M_{rj}^i R_{hkm+}^{r+}\Big|_n + M_{hr}^i R_{jkm+}^{r+}\Big|_n - M_{hj+}^{i+}\Big|_r \left(N_{mk+}^{r+}\Big|_n + R_{nkm}^r\right) = 0 \quad \ldots(4.18)$$

if the space F_n^* is special quasi symmetric and affinely connected.

(g) $$\left(\dot{\partial}_r M_{hj}^i\right) R_{km+}^{r+}\Big|_n - M_{hj+}^{i+}\Big|_r N_{mk+}^{r+}\Big|_n = 0 \quad \ldots(4.19)$$

if the space F_n^* is special quasi symmetric, affinely connected and flat.

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