Indian J. pure appl. Math., 13(9) : 1018–1023, September 1982

TOTALLY REAL SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD WITH SEMI-SYMMETRIC METRIC *F*-CONNECTIONS

R. S. SINHA AND H. C. LAL*
Department of Mathematics, University of Gorakhpur, Gorakhpur (U. P.)

(*Received* 14 *September* 1981)

In a recent paper Yano and Imai (1975) introduced the concept of semi-symmetric metric F-connections in a Kaehlerian manifold. Totally real submanifolds of a Kaehlerian manifold have been studied by Chen and Ogiue (1974), Ludden, Okumara and Yano (1975), Yano (1976) and others. In the present paper we have shown that under certain conditions, totally real submanifolds of a Kaehlerian manifold equipped with semi-symmetric metric F-connections are conformally flat.

1. INTRODUCTION

Let K_{2m} be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighbourhoods $\{U, x^h\}$, where and in the sequel, the indices $h, i, j, k, \ldots,$ run over the range $\{1, 2, 3, \ldots 2m\}$. Let g_{ji}, F_i^h and ∇ denote respectively the Riemannian metric tensor, complex structure tensor and the operator of covariant differentiation with respect to the Christoffel symbols formed with g_{ji}, then we have (Yano 1965, p. 70)

$$\text{(a)} \quad F_i^h F_j^i = -\delta_j^h, \quad \text{(b)} \quad g_{ji} F_r^j F_s^i = g_{rs}, \quad \text{(c)} \quad \nabla_k F_j^i = 0 \qquad \ldots(1.1)$$

In a K_{2m}, the following are also valid

$$\text{(a)} \quad F_h^h = 0 \quad \text{(b)} \quad F_{hj} = -F_{jh} \quad \text{(c)} \quad F_j^h = -F_j^h \quad \text{(d)} \quad F^{hj} = -F^{jh} \ldots(1.2)$$

where $F_{hj} = F_h^r g_{rj}$, $F_j^h = F_{rj} g^{rh}$ and $F^{hj} = F_r^j g^{rh}$.

Let M_n be a n-dimensional totally real submanifold, immersed isometrically in K_{2m}, immersion being represented by $x^h = x^h(u^\alpha)$, where and in the sequel, indices $\alpha, \beta, \gamma, \delta, \ldots$ take the values over the range $\{1, 2, 3, 4, 5, \ldots, n\}$. Let us put $B_\alpha^h = \partial x^h / \partial u^\alpha$ and denote by C_x^h, the $2m - n$ mutually orthogonal unit vectors normal to M_n, where and in the sequel, the indices $x, y, z,$ run over the range $\{(n+1)', (n+2) l, \ldots (2m-n)'\}$, then we have

$$\text{(a)} \quad g_{\alpha\beta} = g_{ji} B_{\alpha\beta}^{ji}, \quad \text{(b)} \quad g_{xy} = g_{ji} C_{xy}^{ji} \qquad \ldots(1.3)$$

where $B_{\alpha\beta}^{ji} = B_\alpha^j B_\beta^i$, $C_{xy}^{ji} = C_x^j C_y^i$ and $g_{\alpha\beta}$, g_{xy} denote respectively the metric tensors

*K. S. Saket P. G. College, Faizabad 224001 (U. P.).

of M_n and the normal bundle. Moreover, for a totally real submanifold M_n of K_{2m}, we have equations of the form (Yano 1976, Imai 1976, Ludden *et al.* 1975)

$$\text{(a)}\quad F_j^h B_\alpha^j = -f_\alpha^x C_x^h, \quad \text{(b)}\quad F_j^h C_y^j = f_y^\alpha B_\alpha^h + f_y^x C_x^h \qquad \ldots(1.4)$$

and consequently we have

$$\left.\begin{array}{l} \text{(a)}\quad F_{ji} B_{\alpha\beta}^{ji} = 0, \quad \text{(b)}\quad F_{ji} B_\alpha^j C_y^i = -f_{\alpha y}, \\ \text{(c)}\ F_{ji} C_{yx}^{ji} = f_{yx}, \quad \text{(d)}\quad F_j^h B_\alpha^j B_h^\beta = 0, \end{array}\right\} \qquad \ldots(1.5)$$

where $B_h^\beta = g^{\beta\alpha} B_\alpha^k g_{kh}$, $f_{\alpha y} = f_\alpha^z g_{zy}$ and $f_{yx} = f_y^z g_{zx}$.

From (1.2b), (1.3a), (1.4a) and (1.4b), we get

$$f_{\alpha y} = f_{y\alpha} \qquad \ldots(1.6)$$

where $f_{y\alpha} = f_y^\beta g_{\beta\alpha}$. Applying the complex structure tensor F to (1.4a) and (1.4b) and using eqns. (1.5) and (1.6), we find

$$\left.\begin{array}{l} \text{(a)}\ f_\alpha^y f_y^\beta = \delta_\alpha^\beta, \quad \text{(b)}\ f_\alpha^y f_y^x = 0, \quad \text{(c)}\ f_y^z f_z^h = 0, \\ \text{(d)}\quad f_y^z f_z^x = -\delta_y^x + f_y^\alpha f_\alpha^x. \end{array}\right\} \qquad \ldots(1.7)$$

Equations (1.7) show that if f_y^x does not vanish, it defines an f-structure in the normal bundle (see Yano 1963).

2. Totally Real Submanifold M_n of K_{2m}

The two special semi-symmetric metric F-connections in a Kaehlerian manifold K_{2m}, introduced by Yano and Imai (1975), which we shall denote by $\overset{1}{\Gamma}{}_{ji}^h$ and $\overset{2}{\Gamma}{}_{ji}^h$ respectively are given by the following equations

$$\overset{1}{\Gamma}{}_{ji}^h = \left\{ {h \atop ji} \right\} + \delta_j^h p_i - g_{ji} p^h + F_j^h q_i + F_i^h q_j - F_{ji} q^h \qquad \ldots(2.1)$$

$$\overset{2}{\Gamma}{}_{ji}^h = \left\{ {h \atop ji} \right\} + \delta_j^h p_i - g_{ji} p^h + F_j^h q_i - F_{ji} q^h \qquad \ldots(2.2)$$

where $p_i = \partial_i p$, p being a function, $p^h = g^{th} p_t$, $q^h = g^{th} q_h$ and

$$q_i = F_{ti} p^t. \qquad \ldots(2.3)$$

Let $\overset{2}{\Gamma}{}_{\beta\gamma}^\alpha$ denote the connection induced from $\overset{2}{\Gamma}{}_{ji}^h$ with respect to C_x^h on a totally real submanifold M_n, immersed isometrically in a Kaehlerian manifold K_{2m}, then we have (Imai (1978) p. 298)

$$\overset{2}{\Gamma}{}_{\beta\gamma}^\alpha = (\partial_\beta B_\gamma^h + \overset{2}{\Gamma}{}_{ji}^h B_{\beta\gamma}^{ji}) B_h^\alpha. \qquad \ldots(2.4)$$

On substituting from (2.4) and using (1.5), we find

$$\overset{2}{\Gamma}{}_{\beta\alpha}^\gamma = \left\{ {\alpha \atop \beta\gamma} \right\} + \delta_\beta^\alpha p_\gamma - g_{\beta\gamma} p^\alpha \qquad \ldots(2.5)$$

where $\left\{ {\alpha \atop \beta\gamma} \right\}$ denote the Christoffel symbols formed with $g_{\alpha\beta}$, $p_\beta = \partial_\beta p$ and $p^\alpha = g^{\alpha\beta} p_\beta$. Now it can be easily seen that $\overset{2}{\Gamma}{}^\alpha_{\beta\gamma}$ is a semi-symmetric connection (1970). Thus, we have

Theorem 2.1—The connection induced on a totally real submanifold M_n of a Kaehlerian manifold K_{2m} with semi-symmetric metric F-connection (2.2) is a semi-symmetric metric connection.

Now, let $\overset{2}{R}{}^h_{kji}$ denote the curvature tensor of the connection $\overset{2}{\Gamma}{}^h_{ji}$, then we have (Yano and Imai 1978)

$$\overset{2}{R}{}^h_{kji} = K^h_{kji} - \delta^h_k p_{ji} + \delta^h_j p_{ki} - p^h_k g_{ji} + p^h_j g_{ki} - F^h_k q_{ji} + F^h_j q_{ki} - q^h_k F_{ji} + q^h_j F_{ki} - 2F_{kj}(p_i q^h - q_i p^h) \quad \ldots(2.6)$$

where K^h_{kji} is the Riemannian curvature tensor of K_{2m} and

$$p_{ji} = \nabla_j p_i - p_j p_i + q_j q_i + \tfrac{1}{2} p_t p^t g_{ji},$$

$$q_{ji} = \nabla_j q_i - p_j q_i - p_i q_j + \tfrac{1}{2} p_t p^t F_{ji},$$

$p^h_k = p_{kt} g^{th}$, $q^h_k = q_{kt} g^{th}$. Evidently we have

$$\text{(a) } p_{jt} = q_{jt} F^t_i, \quad \text{(b) } q_{ji} = -p_{jt} F^t_i \quad \text{(c) } p_{ji} = p_{ij}. \quad \ldots(2.7)$$

Thus, transvecting the covariant form of (2.6) with $B^{kjih}_{\alpha\beta\gamma\delta} = B^k_\alpha B^j_\beta B^i_\gamma B^h_\delta$ and using (1.3a) and (1.5a), we find

$$\overset{2}{R}_{kjih} B^{kjih}_{\alpha\beta\gamma\delta} = K_{kjih} B^{kjih}_{\alpha\beta\gamma\delta} - g_{\alpha\delta} p_{ji} B^{ji}_{\beta\gamma} + g_{\beta\delta} p_{ji} B^{ji}_{\alpha\gamma} - p_{ji} B^{ji}_{\alpha\delta} g_{\beta\gamma} + p_{ji} B^{ji}_{\beta\delta} g_{\alpha\gamma}$$

which with the help of well known Gauss equation

$$K_{\alpha\beta\gamma\delta} = K_{kjih} B^{kjih}_{\alpha\beta\gamma\delta} + H_{\alpha\delta x} H^x_{\beta\gamma} - H_{\beta\delta x} H^x_{\alpha\gamma}$$

for the totally real submanifold of a Kaehlerian manifold, where $H^x_{\alpha\beta}$ are the second fundamental tensor of M_n with respect to the normals C^h_x and $H_{\alpha\beta x} = H^y_{\alpha\beta} g_{yx}$, reduces to the form

$$\overset{2}{R}_{kjih} B^{kjih}_{\alpha\beta\gamma\delta} = K_{\alpha\beta\gamma\delta} - H^x_{\alpha\delta x} H^x_{\beta\gamma} + H_{\beta\delta x} H^x_{\alpha\gamma} - g_{\alpha\delta} p_{ji} B^{ji}_{\beta\gamma} + g_{\beta\delta} p_{ji} B^{ji}_{\alpha\gamma} - p_{ji} B^{ji}_{\alpha\delta} g_{\beta\gamma} + p_{ji} B^{ji}_{\beta\delta} g_{\alpha\gamma}. \quad \ldots(2.8)$$

Now, we assume that

$$\overset{2}{R}{}^h_{kji} = \alpha_{kj} F^h_i \text{ i.e., } \overset{2}{R}_{kjih} = \alpha_{kj} F_{ih} \quad \ldots(2.9)$$

for some tensor α_{kj} in K_{2m} and M^n is totally umblical also, then

$$H^x_{\alpha\beta} = g_{\alpha\beta} H^x \quad \ldots(2.10)$$

where $H^x = (1/n)\ g^{\alpha\beta}\ H^x_{\alpha\beta}$ and consequently

$$H_{\alpha\beta\gamma} = g_{\alpha\beta}\ H_y\ , \qquad ...(2.11)$$

where $H_y = H^x\, g_{xy}$. Substituting from (2.9), (2.10) and (2.11) into (2.8) and using (1.5) we find

$$K_{\alpha\beta\gamma\delta} = (g_{\alpha\delta}\, g_{\beta\gamma} - g_{\beta\delta}\, g_{\alpha\gamma})\, H_x H^x - g_{\alpha\delta}\, p_{ji}\ B^{ji}_{\beta\gamma}$$

$$+\ g_{\beta\gamma}\ p_{ji}\ B^{ji}_{\alpha\gamma} - g_{\beta\delta} p_{ji}\ B^{ji}_{\alpha\delta} + g_{\alpha\gamma}\, p_{ji}\ B^{ji}_{\beta\delta}. \qquad ...(2.12)$$

Transvecting (2.12) with $g^{\beta\gamma}$, we find

$$K_{\alpha\delta} = (n-1)\, g_{\alpha\delta} H_x H^x - g_{\alpha\delta}\, p_{ji}\, B^{ji} - (n-2)\, p_{ji}\, B^{ji} \qquad ...(2.13)$$

where $B^{ji} = B^{ji}_{\alpha\beta}\ g^{\alpha\beta}$. Contracting (2.13) with $g^{\alpha\delta}$ we get

$$p_{ji}\, B^{ji} = -\frac{K}{2\,(n-1)} + \frac{n}{2}\ H_x H^x. \qquad ...(2.14)$$

Substituting from (2.14) into (2.13), we find

$$p_{ji}\, B^{ji}_{\alpha\delta} = \frac{1}{2}\ g_{\alpha\delta} H_x H^x + C_{\alpha\delta}, \qquad ...(2.15)$$

where $C_{\alpha\delta} = -\dfrac{1}{n-2}\left[K_{\alpha\delta} - \dfrac{1}{2\,(n-1)}\ K\, g_{\alpha\delta}\right]$. Substituting from (2.15) into (2.12) we find $C_{\alpha\beta\gamma\delta} = 0$, where $C_{\alpha\beta\gamma\delta}$ is the so called Weyl's conformal curvature tensor of M_n. Thus, we have

Theorem 2.2—Let M_n $(n > 3)$, be a totally umblical, totally real submanifold of a Kaehlerian manifold K_{2m} admitting special semi-symmetric metric F-connection (2.2), whose curvature tensor takes the form (2.9), then M_n is conformally flat.

Yano and Imai (1978) proved the following

Theorem A — Let K_{2m} $(2m \geqslant 4)$ be a Kaehlerian manifold with special semi-symmetric metric F-connection (2.2), whose curvature tensor takes the form (2.9), then K_{2m} is a manifold with vanishing Bochner curvature tensor.

With the help of Theorem 2.2 and Theorem A, we can very easily get the following theorem due to Yano (1976).

Theorem 2.3 — Let M_n $(n \geqslant 4)$ be a totally real, totally umblical submanifold of a Kaehlerian manifold K_{2m} with vanishing Bochner curvature tensor, then, M_n is conformally flat.

Moreover, corresponding results for the connection $\overset{1}{\Gamma}{}^h_{ji}$ have been already obtained by Imai (1978).

3. Totally Real Submanifold M_n of K_{2n}

The present article is devoted to the study of n-dimensional totally real submanifolds of a Kaehlerian manifold K_{2n} admitting special semi-symmetric metric

F-connections. In the present case from eqns. (1.7) we have the following (see Yano 1976)

$$\text{(a)}\quad f_y^x = 0, \quad \text{(b)}\ f_\alpha^y\, f_y^\beta = \delta_\alpha^\beta, \quad \text{(c)}\ f_y^\alpha\, f_\alpha^x = \delta_y^x \qquad \ldots(3.1)$$

and

$$K_{\alpha\beta yx}\, f_\gamma^y\, f_\varepsilon^x = K_{\alpha\beta\gamma\varepsilon}, \qquad \ldots(3.2)$$

where $K_{\alpha\beta\gamma\varepsilon} = K_{\alpha\beta\gamma}^{\rho}\, g_{\rho\varepsilon}$ and $K_{\alpha\beta yx} = K_{\alpha\beta y}^{z}\, g_{zx}$, $K_{\alpha\beta y}^{z}$ being the curvature tensor of the connection induced in the normal bundle. Moreover, equations of Ricci are (1976, p. 357)

$$K_{\alpha\beta yx} = K_{kjih}\, B_{\alpha\beta}^{kj}\, C_{yx}^{ih} - T_{\alpha\beta yx} \qquad \ldots(3.3)$$

where $K_{kjih} = K_{kji}^{t}\, g_{th}$ and $T_{\alpha\beta yx} = H_{\alpha y}^{\varepsilon}\, H_{\beta\varepsilon x} - H_{\beta y}^{\varepsilon}\, H_{\alpha\varepsilon x}$.

Thus, transvecting the covariant form of (2.6) with $B_{\alpha\beta}^{kj}\, C_{yx}^{ih}$ and using (1.2), (1.4), (1.5), (2.7), (3.1) and (3.3), we find

$$\begin{aligned}\overset{2}{R}_{kjih}\, B_{\alpha\beta}^{kj}\, C_{yx}^{ih} = {} & K_{\alpha\beta yx} + T_{\alpha\beta yx} - f_{\alpha x}\, p_{jt}\, B_{\beta\rho}^{jt}\, f_y^\rho \\ & + f_{\beta x} p_{jt}\, B_{\alpha\rho}^{jt}\, f_y^\rho - f_{\beta y}\, p_{jt}\, B_{\alpha\rho}^{jt}\, f_x^\rho \\ & + f_{\alpha y}\, p_{jt}\, B_{\beta\rho}^{jt}\, f_x^\rho . \end{aligned} \qquad \ldots(3.4)$$

Now, let $\overset{2}{R}_{kji}^{h}$ assume the form (2.9) and the second fundamental tensors of M^n with respect to the normals C_x^h and C_y^h commute, i.e. $T_{\alpha\beta yx} = 0$, then equation (3.4) in view of (1.5c) and (3.1a) reduces to the form

$$K_{\alpha\beta yx} = f_{\alpha x}\, p_{jt}\, B_{\beta\rho}^{jt}\, f_y^\rho - f_{\beta x} p_{jt}\, B_{\alpha\rho}^{jt}\, f_y^\rho - f_{\beta y}\, p_{jt}\, B_{\alpha\rho}^{jt}\, f_x^\rho + f_{\alpha y} p_{jt}\, B_{\beta\rho}^{jt}\, f_x^\rho .$$

On contracting the above equation by $f_\gamma^y\, f_\varepsilon^x$ and using (3.1), (3.2), (3.3) together with the fact $f_{\alpha y} = f_{y\alpha}$, we find

$$K_{\alpha\beta\gamma\varepsilon} = g_{\alpha\varepsilon} p_{jt} B_{\beta\gamma}^{jt} - g_{\beta\varepsilon}\, p_{jt}\, B_{\alpha\gamma}^{jt} + g_{\beta\gamma} p_{jt}\, B_{\alpha\varepsilon}^{jt} - g_{\alpha\gamma}\, p_{jt}\, B_{\beta\varepsilon}^{jt} \qquad \ldots(3.5)$$

which on multiplying with $g^{\beta\gamma}$ yields

$$K_{\alpha\varepsilon} = g_{\alpha\varepsilon}\, p_{jt}\, B^{jt} + (n-2)\, p_{jt}\, B^{jt} . \qquad \ldots(3.6)$$

Equation (3.6), in turn, after multiplication with $g^{\alpha\varepsilon}$ gives

$$K = 2\,(n-1)\, p_{jt}\, B^{jt}. \qquad \ldots(3.7)$$

Substituting from (3.7) into (3.6) we find

$$p_{jt}\, B_{\alpha\varepsilon}^{jt} = C_{\alpha\varepsilon}, \qquad \ldots(3.8)$$

where $C_{\alpha\delta} = \frac{1}{n-2}\left[K_{\alpha\delta} - \frac{1}{2(n-1)} K g_{\alpha\delta}\right]$ and consequently the substitution from (3.8) into (3.5) will give $C_{\alpha\beta\gamma\delta} = 0$. Thus, we have

Theorem 3.1—If M_n $(n > 3)$ be totally real submanifold of a Kaehlerian manifold K_{2n} with special semi-symmetric F-connection (2.2) who securvature tensor assumes the form (2.9) and the second fundamental tensors of M_n commute, then M_n is conformally flat.

The above theorem, in view of Theorem (A) immediately gives the following theorem due to Yano (1976).

Theorem 3.2—Let M_n $(n \geqslant 4)$ be a totally real submanifold of a Kaehlerian manifold M_{2n} with vanishing Bochner curvature tensor. If the second fundamental tensors of M_n commute, then M_n is conformally flat.

Following the same process of calculation, for the connection $\overset{1}{\Gamma}{}^h_{ji}$, we can easily prove the following theorem.

Theorem 3.3—If M_n $(n > 3)$ be a totally real submanifold of a Kaehlerian manifold K_{2n} with special semi-symmetric metric F-connection (2.1) whose curvature tensor vanishes and the second fundamental tensor of M_n commute, then M_n is conformally flat.

Finally, it can be remarked that Theorem 3.2 can also be proved with the help of theorem 3.3 and the following theorem due to Yano and Imai (1975, p. 136).

Thorem B—If, in K_{2m} there exists a scalar function p such that the special semi-symmetric metric F-connection (2.1) is of zero curvature, then Bochner curvature tensor of the manifold vanishes.

REFERENCES

Chen, B. Y., and Ogiue, K. (1974). On totally real submanifolds. *Trans. Am. Math. Soc.*, **193**, 257–66.

Imai, T. (1978). Applications of complex and contact conformal connection. *Tensor (N. S.)*, **32**, 297–301.

Ludden, G. D., Okumara, M., and Yano, K. (1975). Totally real submanifolds of a complex manifold. *Atti. Accad. Naz. Lincei Rend. Cl. Sci., Fis. Mat. Natur*, **63**, 346–57.

Yano, K. (1963). On a structure defined by a tensor field of type (1, 1) satisfying $f^3 + f = 0$. *Tensor*, **14**, 99–109.

——— (1965). Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press, Orford, London.

——— (1970). On semi-symmetric metric connections. *Rev. Roum. Math. Pures. Appl.*, **15**, 1579–86.

——— (1976). Totally real submanifolds of a Kaehlerian manifold. *J. Diff. Geom.*, **11**, 351–59.

Yano, K., and Imai, T. (1975). On semi-symmetric metric F-connections. *Tensor (N.S.)*, **29**, 134–38.