Indian J. pure appl. Math., **13**(9) : 1024–1030, September 1982

APPROXIMATION OF ENTIRE HARMONIC FUNCTIONS IN R^3

G. P. KAPOOR AND A. NAUTIYAL

Department of Mathematics, Indian Institute of Technology, Kanpur 208016

(*Received* 22 *August* 1981)

Let H_R, $0 < R < \infty$, be the class of all harmonic functions H in R^3, that are regular in the open ball D_R of radius R centered at the origin and are continuous on the closure $\bar{D}_R$ of D_R. For $H \in H_R$, set

$$E_n(H, R) = \inf_{g \in \pi_n} \{ \max_{(x_1, x_2, x_3) \in D_R} | H(x_1, x_2, x_3) - g(x_1, x_2, x_3) | \}$$

where π_n consists of all harmonic polynomials of degree atmost n. In the present paper, we obtain necessary and sufficient conditions, in terms of the rate of decay of the approximation error $E_n(H, R)$, such that $H \in H_R$ has analytic continuation as an entire harmonic function having finite order and finite type.

1. INTRODUCTION

The harmonic functions in R^3 are the solutions of the Laplace equation

$$\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \frac{\partial^2 H}{\partial x_3^2} = 0. \quad \ldots(1.1)$$

A harmonic function H, regular about the origin, can be expanded as

$$H \equiv H(r, \theta, \varphi) = \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} (a_{nm}^{(1)} \cos m\varphi + a_{nm}^{(2)} \sin m\varphi) P_n^m(\cos \theta) \quad \ldots(1.2)$$

where $x_1 = r \cos \theta$, $x_2 = r \sin \theta \cos \varphi$, $x_3 = r \sin \theta \sin \varphi$ and $P_n^m(t)$ are associated Legendre's functions of first kind of degree m and order n. A hormonic polynomial of degree k is a polynomial of degree k in x_1, x_2 and x_3 which satisfies (1.1).

A harmonic function H is said to be regular in $D_R = \left\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < R^2 \right\}$, $0 < R \leqslant \infty$, if the series (1.2) converges uniformly on compact subsets of D_R. A harmonic function H is called entire if it is regular in D_∞.

The order ρ and type T of an entire harmonic function H are defined as

$$\rho \equiv \rho(H) = \limsup_{r \to \infty} \frac{\log \log M(r, H)}{\log r} \quad \ldots(1.3)$$

$$T \equiv T(H) = \limsup_{r \to \infty} \frac{\log M(r, H)}{r^\rho}, \quad 0 < \rho < \infty \quad \ldots(1.4)$$

where

$$M(r, H) = \max_{x_1^2 + x_2^2 + x_3^2 = r^2} | H(x_1, x_2, x_3) | .$$

Recently, Fryant (1978) related ρ and T of an entire harmonic function H with the rate of decrease of coefficients $a_{nm}^{(i)}$ in (1.2), $i = 1, 2$. Analogous results for the solutions of (1.1) which depend only on the variables $x = x_1$ and $y = (x_2^2 + x_3^2)^{1/2}$ have been found in Fryant (1977) and Gilbert (1969, Theorem 4.3.4).

Let H_R, $0 < R < \infty$, denote the class of all harmonic functions H regular in D_R and continuous on $\bar{D}_R$, the closure of D_R. For $H \in H_R$, let $E_n(H, R)$, the error in approximating the function H by harmonic polynomials of degree at most n in uniform norm, be defined as

$$E_n(H, R) = \inf_{g \in \pi_n} \| H - g \|_R \qquad \ldots(1.5)$$

where π_n consists of all harmonic polynomials of degree atmost n and

$$\| H - g \|_R = \max_{(x_1, x_2, x_3) \in \bar{D}_R} | H(x_1, x_2, x_3) - g(x_1, x_2, x_3) | .$$

In the present paper, we determine the necessary and sufficient conditions on the rate of decrease of $E_n(H, R)$, as $n \to \infty$, such that $H \in H_R$ has analytic continuation as an entire harmonic function having growth parameters ρ and T, defined by (1.3) and (1.4). Analogous results for the solutions of (1.1), which depend only on the variables $x = x_1$ and $y = (x_2^2 + x_3^2)^{1/2}$, have been obtained by McCoy (1979).

We prove :

Theorem 1 — Let $H \in H_R$. Then, H has analytic continuation as an entire harmonic function, if and only if,

$$\limsup_{n \to \infty} (E_n(H, R))^{1/n} = 0. \qquad \ldots(1.6)$$

Theorem 2 — Let $H \in H_R$. Then, H has analytic continuation as an entire harmonic function of finite order ρ, if and only if,

$$\limsup_{n \to \infty} \frac{n \log n}{-\log E_n(H, R)} \equiv J(H) < \infty. \qquad \ldots(1.7)$$

Further, $J(H) = \rho$ holds.

Theorem 3—Let $H \in H_R$. Then H has analytic continuation as an entire harmonic function of order ρ ($0 < \rho < \infty$) and type T ($0 < T < \infty$), if and only if,

$$\limsup_{n \to \infty} n (E_n(H, R))^{\rho/n} \equiv J^*(H) \qquad \ldots(1.8)$$

satisfies $0 < J^*(H) < \infty$. Further

$$J^*(H) = e \rho T R^\rho$$

holds.

2. Preparatory Lemmas

In this section we give some lemmas that are used in proving Theorems 1, 2 and 3.

Lemma 1 — Associated Legendre's functions $P_n^m(t)$ satisfy

$$\max_{-1\leqslant t\leqslant 1} |P_n^m(t)| \leqslant K[(n+m)!/(n-m)!]^{1/2}, \qquad \ldots(2.1)$$

where K is a constant independent of n and m.

Proof : It is known (Erdélyi *et al.* 1953, p. 148) that

$$P_n^m(t) = \frac{(1-t^2)^{m/2}}{2^n\, n!} \frac{d^{n+m}(t^2-1)^n}{d\,t^{n+m}}.$$

Thus, $(P_n^m(t))^2$ is a polynomial of degree $2n$. Now, using Theorem 2.2.1 of Sewell (1942) we obtain

$$\max_{-1\leqslant t\leqslant 1} |P_n^m(t)|^2 \leqslant K'\, 2n \int_{-1}^{1} (P_n^m(t))^2\, dt \qquad \ldots(2.2)$$

where K' is a constant independent of n and m. It is also known (Sansonne 1959, p. 247) that

$$\int_{-1}^{1} (P_n^m(t))^2\, d\,t = \frac{2}{2\,n+1} \frac{(n+m)!}{(n-m)!}. \qquad \ldots(2.3)$$

Combining (2.2) and (2.3) we get (2.1). This proves the lemma.

Lemma 2 — Let $H \in H_R$ be entire and $r' > 1$. Then, for all $r > 2r'R$ and all sufficiently large values of n, we have

$$E_n(H, R) \leqslant \bar{K}\, M(r, H)\, (r'R/r)^{n+1}.$$

Here $\bar{K}$ is a constant.

Proof : Let

$$h_n = \sum_{k=0}^{n} r^k \sum_{m=0}^{k} (a_{km}^{(1)} \cos m\varphi + a_{km}^{(2)} \sin m\varphi)\, P_k^m(\cos\theta).$$

Then $h_n \in \pi_n$. Now, using (1.5) and Lemma 1 we get

$$E_n(H, R) \leqslant \| H - h_n \|_R$$

$$\leqslant \sum_{k=n+1}^{\infty} R^k \left| \sum_{m=0}^{k} (a_{km}^{(1)} \cos m\varphi + (a_{km}^{(2)} \sin m\varphi)\, P_k^m(\cos\theta) \right|$$

$$\leqslant K \sum_{k=n+1}^{\infty} R^k (2\,k+1) \max_{m,i} \left[\left| a_{km}^{(i)} \right| \left(\frac{(k+m)!}{(k-m)!} \right)^{1/2} \right]. \qquad \ldots(2.4)$$

For an entire harmonic function H we have [Fryant 1978, (2.3)]

$$\max_{m,i} \left[|a_{km}^{(i)}| \left(\frac{(k+m)!}{(k-m)!} \right)^{1/2} \right] \leqslant 2\sqrt{(2\,k+1)}\; M(r, H)/r^k. \qquad \ldots(2.5)$$

Combining (2.4) and (2.5) we get

$$E_n(H, R) \leqslant 2\,K\,M(r, H) \sum_{k=n+1}^{\infty} (2\,k+1)^{3/2} (R/r)^k.$$

Since $(2k+1)^{3/2k} \to 1$ as $k \to \infty$ and $r' > 1$, we get $(2k+1)^{3/2} < (r')^k$ for $k > k_0 = k_0(r')$. Thus, for $n > k_0$, the above inequality gives

$$E_n(H, R) \leqslant 2K\,M(r, H) \sum_{k=n+1}^{\infty} (r'\,R/r)^k$$

$$= 2\,K\,M(r, H)\ \frac{(r'\,R/r)^{n+1}}{1-(r'\,R/r)}\,.$$

The lemma now follows from the above inequality since $r > 2\,r'\,R$.

Lemma 3—Let $H \in H_R$. Then, for any $R_* < R$ and $n \geqslant 1$, we have

$$R_*^n \max_{m,i} \left[\ |\,a_{nm}^{(i)}\,|\ \left(\frac{(n+m)\,!}{(n-m)\,!}\right)^{1/2}\right] \leqslant K_0\,(2\,n+1)\,E_{n-1}\,(H, R).$$

K_0 is a constant.

PROOF: For $H \in H_R$, it is known Fryant (1978) that

$$a_{nm}^{(i)}\ R_*^n = \frac{2\,n+1}{2\,\pi\,\alpha_m}\ \frac{(n-m)\,!}{(n+m)\,!} \int_0^{2\pi}\int_0^{2\pi} H\,(R_*, \theta, \varphi)$$

$$\times\ P_n^m(\cos\theta)\ T_{i,m}(\varphi)\ \sin\theta\ d\theta\ d\varphi$$

for every $R_* < R$. Here $\alpha_m = 2$ if $m=0$ and $\alpha_m = 1$ otherwise, $T_{1,m}(\varphi) = \cos m\varphi$, $T_{2,m}(\varphi) = \sin m\varphi$. Thus, for any $g \in \pi_{n-1}$ we get

$$a_{nm}^{(i)}\ R_*^n = \frac{2\,n+1}{2\,\pi\,\alpha_m}\ \frac{(n-m)\,!}{(n+m)\,!} \int_0^{2\pi}\int_0^{\pi} (H\,(R_*, \theta, \varphi) - g\,(R_*, \theta, \varphi))$$

$$\times\ P_n^m\ (\cos\theta)\ T_{i,m}\ (\varphi)\ \sin\theta\ d\theta\ d\varphi.$$

Now, using Lemma 1, we get

$$|\ a_{nm}^{(i)}\ |\ R_*^n \leqslant (2\,n+1)\,K\,\pi \left(\frac{(n-m)\,!}{(n+m)\,!}\right)^{1/2} \|\,H - g\,\|_R. \qquad \ldots(2.6)$$

By the definition of $E_n(H, R)$ there exists $\tilde{g} \in \pi_{n-1}$ such that

$$\|\,H - \tilde{g}\,\|_R \leqslant 2\,E_{n-1}\,(H, R). \qquad \ldots(2.7)$$

Taking, in particular, $g = \tilde{g}$ in (2.6) and then using (2.7) the lemma follows from (2.6).

3. PROOFS OF THEOREMS 1, 2 AND 3

Proof of Theorem 1—First suppose that H is entire. Then, it follows from Lemma 2 that

$$\limsup_{n \to \infty} \quad (E_n (H, R))^{1/n} \leqslant r' R/r$$

for all $r > 2 r' R$. This easily gives

$$\limsup_{n \to \infty} \quad (E_n (H, R))^{1/n} = 0.$$

This proves the necessity part.

On the other hand, for $H \in H_R$, using (1.2), Lemma 1 and Lemma 3 we get

$$\left| \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} (a_{nm}^{(1)} \cos m\varphi + a_{nm}^{(2)} \sin m\varphi) P_n^m (\cos \theta) \right|$$

$$\leqslant | a_{00}^{(1)} | + K \sum_{n=1}^{\infty} (2n + 1) r^n \max_{m, i} \left[| a_{nm}^{(i)} | \left(\frac{(n+m)!}{(n-m)!} \right)^{1/2} \right]$$

$$\leqslant | a_{00}^{(1)} | + K K_0 \sum_{n=1}^{\infty} (2n + 1)^2 E_{n-1} (H, R) (r/R_*)^n, \qquad \ldots(3.1)$$

for some $R_* < R$. Now if (1.6) holds then it follows from (3.1) that the series on the right hand side of (1.2) converges uniformly on compact subsets of D_∞ and so H is entire.

This proves the theorem.

Proof of Theorem 2 — (i) First suppose that H is an entire harmonic function of finite order ρ and $\rho < \rho_0$. Now, using Lemma 2, we have

$$\log E_n (H, R) + (n + 1) \log (r/r' R)) \leqslant \log \bar{K} + r^{\rho_0} \qquad \ldots(3.2)$$

for all sufficiently large values of r and n. Taking, in particular, $r = r(n) = r' R (n)^{1/\rho_0}$ in (3.2) we get

$$\log E_n (H, R) + \frac{n+1}{\rho_0} \log n \leqslant \log \bar{K} + n (r' R)^{\rho_0}$$

for all sufficiently large values of n. The above relation easily gives

$$J (H) \leqslant \rho. \qquad \ldots(3.3)$$

On the other hand, using (3.1), we get

$$M (r, H) \leqslant | a_{00}^{(1)} | + K K_0 \sum_{n=1}^{\infty} (2n + 1)^2 E_{n-1} (H, R) (r/R_*)^n$$

$$= | a_{00}^{(1)} | + K K_0 \, m (r, h) \qquad \ldots(3.4)$$

where

$$h (z) = \sum_{n=1}^{\infty} (2n + 1)^2 E_{n-1} (H, R) (z/R_*)^n, \qquad \ldots(3.5)$$

by Theorem 1, is an entire function of a single complex variable z and

$$m (r, h) = \max_{|z| \leqslant r} | h (z) | .$$

Using (3.4) and applying the formula expressing the order of an entire function of a single complex variable in terms of its Taylor coefficients (Boas 1954, p. 9) to the function $h(z)$ we obtain

$$\rho \leqslant J(H). \qquad \ldots(3.6)$$

Combining (3.3) and (3.6) we get the necessity part of the theorem.

(ii) Conversely, suppose (1.7) holds. Then

$$\limsup_{n \to \infty} (E_n(H, R))^{1/n} = 0$$

and so, by Theorem 1, H is entire. Sufficiency part now follows from the necessity part.

This completes the proof of the theorem.

Proof of Theorem 3—(i) First, let H be an entire harmonic function of order ρ and type T. Let $T < T_0$. Then using Lemma 2, we have

$$\log E_n(H, R) + (n+1) \log (r/r' R) \leqslant \log \bar{K} + T_0 r^\rho \qquad \ldots(3.7)$$

for all sufficiently large values of r and n. Choosing, in particular, $r = r(n) = (n/\rho T_0)^{1/\rho}$ in (3.7) we get

$$\log E_n(H, R) + \frac{n+1}{\rho} \log \left[\frac{n}{\rho T_0} (r' R)^{-\rho}\right] \leqslant \log \bar{K} + \frac{n}{\rho}$$

for all sufficiently large values of n. The above relation gives that

$$\limsup_{n \to \infty} n (E_n(H, R))^{\rho/n} \leqslant e \rho T R^\rho \qquad \ldots(3.8)$$

since $r' > 1$ and $T_0 > T$ are arbitrary.

It follows from Theorem 2 and Boas (1954, p. 9) that the entire function $h(z)$, given by (3.5), is of order ρ. Now using (3.4) and applying the formula expressing the type of an entire function of a single complex variable in terms of its Taylor coefficients (Boas 1954, p. 11) to the function $h(z)$ we get

$$\limsup_{n \to \infty} n (E_n(H, R))^{\rho/n} \geqslant e \rho T R^\rho, \qquad \ldots(3.9)$$

since $R_* < R$ is arbitrary. Necessity part of the theorem now follows from (3.8) and (3.9).

(ii) Conversely, suppose that (1.8) holds. Then it follows that

$$\limsup_{n \to \infty} \frac{n \log n}{-\log E_n(H, R)} = \rho$$

and so, by Theorem 2, H is an entire harmonic function of order ρ. Sufficiency part now follows from the necessity part.

This proves the theorem.

References

Boas, R. P. (Jr.) (1954). Entire functions. Academic Press, New York.

Erdélyi, A., *et al.* (1953). Higher Transcendental Functions, Vol. I. McGraw-Hill Book Co., Inc., New York.

Fryant, A. J. (1977). Growth and complete sequences of generalized axisymmetric potentials. *J. Approx. Theory*, 19, 361–70.

———(1978). Growth of entire harmonic functions in R^3. *J. Math. Anal. Appl.*, **66**, 599–605.

Gilbert, R. P. (1969). Function theoretic methods in partial differential equations. Academic Press, New York.

McCoy, P. A. (1979). Polynomial approximation and growth of generalized axisymmetric potentials. *Canad. J. Math.*, **31**, 49–59.

Sansonne, G. (1959). Orthogonal Functions. Interscience, New York.

Sewell, W. E. (1942). Degree of approximation by polynomials in the complex domain. Ann. of Math. Studies No. 9, Princeton University Press, Princeton.