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ON GENERALISED NÖRLUND METHODS OF SUMMABILITY-II

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The object of this paper is to establish some relations between the products of two generalised Nörlund methods (N, r, α) and (N, p, α) (N, q, α). Our theorems obtained here generalized theorems given by Das (1968a), which state the inclusion relations between the products of two Nörlund methods (N, r) and (N, p) (N, q).

1. DEFINITIONS AND NOTATIONS

Let $p = \{p_n\}$ and $\alpha = \{\alpha_n\}$ be sequences of real or complex numbers such that $\alpha_n \neq 0$ for $n \geqslant 0$ and $\alpha_n = 0$, $p_n = 0$ $n < 0$.

Let $\sum\limits_{n=0}^{\infty} a_n$ be an infinite series with $\{s_n\}$ as the sequence of its partial sums. Also let (N, p, α) denotes the generalised Nörlund method in which the sequence $\{s_n\}$ is transformed into $\left\{t^p_{\alpha,n}\right\}$ given by

$$t^p_{\alpha,n} = \frac{1}{(p * \alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu}\, \alpha_\nu\, s_\nu;$$

where

$$(p * \alpha)_n = \sum_{\nu=0}^{n} p_{n-\nu}\, \alpha_\nu \neq 0 \text{ for } n \geqslant 0.$$

If $t^p_{\alpha,n} \to s$ (finite) as $n \to \infty$, then the series $\sum\limits_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is said to be summable by the generalised Nörlund method (N, p, α) to 's' (see Borwein 1958). We denote it by $\sum\limits_{n=0}^{\infty} a_n = s\ (N, p, \alpha)$ or $s_n \to s\ (N, p, \alpha)$. The method (N, p, α) is said to be regular if it sum every convergent series to its ordinary sum. The necessary and sufficient conditions for the regularity of (N, p, α) are

(i) $\sum\limits_{\nu=0}^{n} | p_{n-\nu}\, \alpha_\nu | = O\,[\, | (p * \alpha)_n | \,]\ (n \geqslant 0)$

and

(ii) $p_{n-\nu} = O\,[(p * \alpha)_n]$ as $n \to \infty$ (ν-fixed).

This follows from Toeplitz's theorem (see Hardy 1949, Theorem 2). The method (N, p, α) reduces to the Nörlund method (N, p) when $\alpha_n = 1$ for all n (Hardy 1949, p. 64); and to the Riesz method $(\bar{N}, \alpha)$ when $p_n = 1$ for all n (Hardy 1949, p. 57).

For two given summability methods A and B we use $A \subset B$ to mean that every series summable A to s (finite) is also summable B to s (finite).

The method (N, p, α) will be positive if $p_n \geqslant 0$ and $\alpha_n \geqslant 0$ for $n \geqslant 0$.

Given any sequence $\{p_n\}$, we write

$$p(z) = \sum_{n=0}^{\infty} p_n z^n \, ; \, p_{-m} = 0 \ (m > 0)$$

and

$$[p(z)]^{-1} = \sum_{n=0}^{\infty} c_n z^n$$

whenever the series on the right converges.

We define the sequence $r = \{r_n\}$ of constants by means of the formal identity

$$r(z) = p(z) \, . \, q(z); \; r_{-m} = 0.$$

(Since the possibility that the series defining one or more of $p(z)$, $q(z)$ and $r(z)$ might have zero radius of convergence has not been excluded. This relation is to be regarded as very formal.)

As usual we say that the sequence $\{p_n\} \in M$, If

$$p_0 = 1, p_n > 0, \ \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1 \text{ for } n = 0, 1, 2, \dots .$$

Throughout this paper, we write, for any sequence 'f_n' and for an integer h;

$$\delta f_n = f_n - f_{n-1}$$

$$\Delta f_n = f_n - f_{n+1}$$

$$\Delta^0 f_n = f_n = \delta^0 f_n$$

$$f_n^{(h)} = f_0^{(h-1)} + f_1^{(h-1)} + \dots f_n^{(h-1)}$$

$$f_n^{(0)} = f_n$$

$$F_n = f_n^{(1)} = \sum_{\nu=0}^{n} f_\nu .$$

For h and l (non-negative integers) we have

$$\Delta^h \Delta^l (f_n) = \Delta^{h+l} f_n$$

and

$$\Delta^h (f_n g_n) = \sum_{j=0}^{h} \binom{h}{j} \Delta^j (f_n) \Delta^{h-j} (g_{n+j})$$

$$= \sum_{j=0}^{h} h_{c_j} \Delta^j (f_n) \Delta^{h-j} (g_{n+j}) .$$

The capital letters C and H are to denote constants, not necessarily to be same at each occurence.

Let $\left\{ N^{p,q}_{\alpha,n} \right\}$ denotes the (N, p, α)-transform of (N, q, α)-transform of $\{s_n\}$ and the corresponding summability by (N, p, α) (N, q, α).

Das (1968a) has considered the most general problem of relative effectiveness of (N, p) (N, q) and (N, r) methods and has established various inclusion and equivalence theorems.

In this present paper, we investigate relations between the methods (N, p, α) (N, q, α) and (N, r, α). Our theorems obtained here generalized some the theorems of Das (1968a).

2. Inclusion theorems

Theorem 1 — The necessary and sufficient conditions that

$$(N, r, \alpha) \subset (N, p, \alpha)\,(N, q, \alpha) \quad \ldots(2.1)$$

are

$$\sum_{k=0}^{n} |\lambda_{n,k}| = O(1)\ (n \geqslant 0) \quad \ldots(2.2)$$

and

$$\lambda_{n,k} = o(1) \text{ as } n \to \infty \ (k\text{-fixed}) \quad \ldots(2.3)$$

$$\lambda_{n,k} = \begin{cases} \dfrac{(r * \alpha)_n}{(p * \alpha)_n} \displaystyle\sum_{v=k}^{n} \dfrac{p_{n-v}\,\alpha_v\,c_{v-k}}{(q * \alpha)_v} = (k \leqslant n) \\ 0\ (k > n). \end{cases} \quad \ldots(2.4)$$

Further let the method (N, p, α) be regular; $c_n = O(1)$; $\alpha_n > 0$; (2.2) holds; and either

(a) $\left\{\dfrac{(q * \alpha)_n}{\alpha_n}\right\} \to Q \neq 0$ and $\dfrac{1}{\alpha_n} = O(1)\ (n \geqslant 0)$

(or)

(b) $|(q * \alpha)_n| \to \infty$

Then (2.1) holds.

The case in which $\alpha_n = 1$ for all n, is given by Das (1968a, Theorem 1). In the case in which $\alpha_n = 1$ (all n), it was pointed out by Das (1968a) that if (N, q) is either positive or regular, then either (a) or (b) must hold. This does not apply in the general case, more strongly, we can have (N, q, α) is both positive and regular but still need not have either (a) or (b) holding. Here is an example. Take

$$\alpha_{2n} = 1;$$

$$\alpha_{2n+1} = 2;$$

$$q_n = 2^{-n}.$$

Then

$$(q * \alpha)_{2n} = \sum_{\nu=0}^{n} q_{2\nu} + 2 \sum_{\nu=0}^{n-1} q_{2\nu+1}$$

$$= \sum_{\nu=0}^{n} 2^{-2\nu} + \sum_{\nu=0}^{n-1} 2^{-2\nu}$$

$$= \frac{4}{3} (1-2^{-2n-2}) + \frac{4}{3} (1-2^{-2n})$$

$$\to \frac{8}{3} \text{ as } n \to \infty.$$

Also

$$(q * \alpha)_{2n+1} = 2 \sum_{\nu=0}^{n} q_{2\nu} + \sum_{\nu=0}^{n} q_{2\nu+1}$$

$$= \frac{5}{2} \sum_{\nu=0}^{n} 2^{-2\nu}$$

$$= \frac{10}{3} (1-2^{-2n-2})$$

$$\to \frac{10}{3} \text{ as } n \to \infty.$$

Thus, as $n \to \infty$

$$\frac{(q * \alpha)_{2n}}{\alpha_{2n}} \to \frac{8}{3}\, ; \; \frac{(q * \alpha)_{2n+1}}{\alpha_{2n+1}} \to \frac{5}{3}$$

so that neither (*a*) nor (*b*) holds. But clearly, for fixed ν,

$$\frac{q_{n-\nu}}{(q * \alpha)_n} = o\,(1)$$

as $n \to \infty$, so that (N, q, α) is regular.

Since the conditions of the remark are only sufficient and 'not necessary' it is worth mentioning that we can get an example in which (N, p, α) is regular, $c_n = O\,(1)$ and (N, q, α) is both regular and positive but in which not only are (*a*) and (*b*) both false, but it is false that (2.3) holds. Let q, α be as before, and take

$$p_n = 1 \; (n = 0, 1)$$

$$p_n = 0 \; (n \geqslant 2)$$

Then $c_n = (-1)^n$

so that $c_n = O\,(1)$; also it is easily verified that (N, p, α) is regular.

Now $(p * \alpha)_n = 3$ (all $n \geqslant 1$). Since for fixed k, $(r * \alpha)_n \neq 0$ and (2.3) is equivalent to

$$\sum_{\nu=k}^{n} \frac{p_{n-\nu}\, \alpha_\nu\, c_{\nu-k}}{(q * \alpha)_\nu} = o\,(1). \qquad \ldots(2.5)$$

But, a straight forward calculation shows that the expression on the left of (2.5) tends to $\frac{9}{40}(-1)^{k+1}$ as $n \to \infty$.

Theorem 2—If

(i) $\alpha_n > 0$; $q_n \geqslant 0$ and for all v, $(q * \alpha)_v \geqslant c$ (a positive constant)

(ii) $\{p_n\} \in M$

(iii) (N, p, α) is regular, which is equivalent to $p_n = o((p * \alpha)_n)$

(iv) $\left\{ \frac{(q * \alpha)_n}{\alpha_n} \right\}$ is non-decreasing

hold; then (2.1) holds.

The case in which $\alpha_n = 1$ for all n; is given by Das (1968a, Theorem 2).

Theorem 3 — If

(i) $\alpha_0 > 0$ and $\{\alpha_n\}$ is non-decreasing

(ii) $\{\delta^h p_n\} \in M$; h-being non-negative integer

(iii) $q_n \geqslant 0$ and $\Delta^{k+1}\left[\frac{\alpha_n}{(q * \alpha)_n}\right] \geqslant 0$ $(k = 0, 1, 2, \ldots, h)$

hold; then (2.1) holds.

For $h = 0$ we get a restricted result of Theorem 2 and for $\alpha_n = 1$ for all n we get the result (Das 1968a, Theorem 3).

3. The Lemmas

Lemma 1 (Das 1968a, Lemma 1)—Let $\{\delta^h p_n\} \in M$; where h is a non-negative integer

Then

$$c_o^h > 0;\ c_n^{(h)} \leqslant 0\ (n = 1, 2, \ldots)$$

$$c_n^{(h+1)} \geqslant 0$$

and

$$\sum_{n=0}^{\infty} | c_n^{(h)} | < \infty$$

Lemma 2 (Das 1968a, Lemma 2)—Let h be non-negative integer and suppose that $\{\delta^h p_n\} \in M$. Then for $k \leqslant v \leqslant n$ we have

$$0 \leqslant \sum_{u=k}^{v} \left(\Delta_u^h p_{n-u} \right) c_{u-k}^{(h)} \leqslant \left(\Delta_k^h p_{n-k} \right) c_{v-k}^{(h+1)}.$$

Lemma 3 (Das 1968a, Lemma 3)—Let $\{\delta^h p_n\} \in M$. Then

$$p_n > 0;\ (n + h + 1)p_n \leqslant (h + 1)P_n.$$

In particular (N, p) is regular.

Using this lemma we prove the following more result.

Lemma 4 — Let

(i) $\alpha_0 > 0$; $\{\alpha_n\}$ is non-decreasing.

(ii) $\{\delta^h p_n\} \in M$; h being non-negative integer.

Then $p_n \geqslant 0$ and

$$\alpha_0 (n + h + 1)p_n \leqslant (h + 1)(p * \alpha)_n$$

In particular (N, p, α) is regular.

PROOF : By Lemma 3, we have

$$\alpha_0 (n + h + 1)p_n \leqslant \alpha_0 (h + 1)P_n.$$

But since $p_n \geqslant 0$ and $\{\alpha_n\}$ is non-decreasing, we have

$$\alpha_0 P_n \leqslant \alpha_0 p_n + \alpha_1 p_{n-1} + \ldots \alpha_n p_0 = (p * \alpha)_n$$

and the results follows.

4. PROOFS OF THEOREMS 1—3

The important case in dealing with the problem of this nature is that in which (N, p, α) and (N, q, α) are regular, so if in proving any theorem, we find it convenient to assume the regularity, we may do so without any serious loss of generality. To a lessor extend a similar remark applies to the assumption that (N, p, α) or (N, q, α) is positive.

Proof of Theorem 1

By definition

$$t^{p}_{\alpha, n} = \frac{1}{(p * \alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu}\, \alpha_\nu\, s_\nu \qquad \ldots(4.1)$$

and

$$t^{r}_{\alpha, n} = \frac{1}{(r * \alpha)_n} \sum_{\nu=0}^{n} r_{n-\nu}\, \alpha_\nu\, s_\nu$$

$$= \frac{1}{(r * \alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu}\, (q * \alpha)_\nu\, t^{q}_{\alpha, \nu} \qquad \ldots(4.2)$$

using inversion formula

$$t^{q}_{\alpha, n} = \frac{1}{(q * \alpha)_n} \sum_{\nu=0}^{n} c_{n-\nu}\, (r * \alpha)_\nu\, t^{r}_{\alpha, \nu}\,. \qquad \ldots(4.3)$$

Again by definition

$$t^{p, q}_{\alpha, n} = \frac{1}{(p * \alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu}\, \alpha_\nu\, t^{q}_{\alpha, \nu}$$

(*equation continued on p.* 1037)

$$= \frac{1}{(p * \alpha)_n} \sum_{\nu=0}^{n} \frac{p_{n-\nu}\, \alpha_\nu}{(q * \alpha)_\nu} \sum_{k=0}^{\nu} c_{\nu-k}\, (r * \alpha)_k\, t^r_{\alpha,k};$$

(using (4.3))

$$= \sum_{k=0}^{n} \lambda_{n,k}\, t^r_{\alpha,k}\text{; where } \lambda_{n,k} \text{ is given by (2.4).}$$

Further, If $s_n = 1$ for all n, then $t^p_{\alpha,n} = 1$ and $t^q_{\alpha,n} = 1$ for all n and this implies that $t^r_{\alpha,n} = 1$ for all n. Therefore

$$\sum_{k=0}^{n} \lambda_{n,k} = 1.$$

Hence the conditions (2.2) and (2.3) are necessary and sufficient for the truth of (2.1); by Toeplitz's theorem, (Hardy 1949a. Theorem 2).

Next we consider the case (a), the hypothesis implies that $\left\{\frac{\alpha_n}{(q * \alpha)_n}\right\} \to \frac{1}{Q} \neq 0.$ From (2.4) we have

$$\lambda_{n,k} = \frac{(r * \alpha)_k}{Q} \sum_{\nu=k}^{n} \frac{p_{n-\nu}\, c_{\nu-k}}{(p * \alpha)_n}$$

$$+ \frac{(r * \alpha)_k}{(p * \alpha)_n} \sum_{\nu=k}^{n} p_{n-\nu}\, c_{\nu-k} \left[\frac{\alpha_\nu}{(q * \alpha)_\nu} - \frac{1}{Q}\right]$$

$$= \Sigma_1 + \Sigma_2 \text{ (say).}$$

Now $\Sigma_1 = 0$ for $n > k$ and $\Sigma_2 = o(1)$ as $n \to \infty$ and k fixed, $\Big($ since (N, p, α) is regular and $\frac{c_{\nu-k}}{\alpha_\nu} = O(1)$ and this will further imply $\left[\frac{\alpha_\nu}{(q * \alpha)} - \frac{1}{Q}\right]\left[\frac{c_{\nu-k}}{\alpha_\nu}\right] = o(1)$ as $\nu \to \infty$ and k fixed$\Big)$. Hence $\lambda_{n,k} = o(1)$ as $n \to \infty$ and k fixed.

For case (b), the hypothesis implies $\frac{c_{\nu-k}}{(q * \alpha)_\nu} = o(1)$ as $\nu \to \infty$. Also (N, p, α) is regular. Therefore we have $\lambda_{n,k} = o(1)$ as $n \to \infty$ and for k-fixed.
This completes the proof of Theorem 1.

Corollary 1 — Let $\alpha_n > 0$; $q_n \geqslant 0$; $(\overline{N}, \alpha)$ is regular and $\left\{\frac{(q * \alpha)_n}{\alpha_n}\right\}$ is non-decreasing. Then $(N, Q, \alpha) \subset (\overline{N}, \alpha)\,(N, q, \alpha)$. This follows from Theorem 1 on putting $p_n = 1$ for all n. In this case.

$c_0 = 1$, $c_1 = -1$ and $c_n = 0$ for $n > 1$. and so for $k < n$

$$\lambda_{n,k} = \frac{(1 * q * \alpha)_k}{(1 * \alpha)_n} \left[\frac{\alpha_k}{(q * \alpha)_k} - \frac{\alpha_{k+1}}{(q * \alpha)_{k+1}}\right].$$

For $k = n$, only last term is not there in the bracket. Thus $\lambda_{n,k} \geqslant 0$ (since $\left\{ \frac{(q * \alpha)_n}{\alpha_n} \right\}$ is non-decreasing). $(\bar{N}, \alpha)$ is regular, implies $(1 * \alpha)_n \to \infty$ as $n \to \infty$. Hence $\lambda_{n,k} = o(1)$ as $n \to \infty$ and for k fixed. The required inclusion thus follows. putting $\alpha_n = 1$ for all n in the corollary we obtain Corollary to Theorem 1 of Das (1968a).

Proof of Theorem 2

The condition (ii) of hypothesis implies $\sum_{n=0}^{\infty} | c_n | < \infty$ which further implies $c_n = o(1)$ as $n \to \infty$. It therefore follows from condition (i) of the hypothesis that

$$\frac{c_{\nu-k}}{(q * \alpha)_\nu} = o(1) \text{ as } \nu \to \infty \text{ and } k\text{-fixed.}$$

Therefore (2.3) holds (since (N, p, α) is regular).
We have only to show (2.2) now,
From (2.4) it is evident that $\lambda_{n,n} > 0$ so we consider $\lambda_{n,k}$ for $k < n$. Using Abel's transformation

$$\lambda_{n,k} = \frac{(r * \alpha)_k}{(p * \alpha)_n} \sum_{\nu=k}^{n} \frac{p_{n-\nu}\, \alpha_\nu\, c_{\nu-k}}{(q * \alpha)_\nu}.$$

$$= \frac{(r * \alpha)_k}{(p * \alpha)_n} \sum_{\nu=k}^{n} \Delta \left[\frac{\alpha_\nu}{(q * \alpha)_\nu} \right] \sum_{u=k}^{\nu} p_{n-u}\, c_{u-k}$$

$$+ \frac{(r * \alpha)_k}{(p * \alpha)_n} \frac{\alpha_{n+1}}{(q * \alpha)_{n+1}} \sum_{u=k}^{n} p_{n-u}\, c_{u-k} \qquad \ldots(4.5)$$

$$= \lambda^{(1)}_{n,k} + \lambda^{(2)}_{n,k} \text{ (say).}$$

But $\lambda^{(2)}_{n,k} = 0$ for $n > k$.

Also by Lemma 2 (for $h = 0$) and by condition (iv) of the hypothesis imply $\lambda^{(1)}_{n,k} \geqslant 0$ for $k < n$.
Thus we have $\lambda_{n,k} \geqslant 0$ for $k \leqslant n$ and —

$$\sum_{k=0}^{n} | \lambda_{n,k} | = \sum_{k=0}^{n} \lambda_{n,k} = 1.$$

The required inclusion now follows from Theorem 1.

Proof of Theorem 3

Now $\{\delta^h p_n\} \in M$ implies $\sum_{n=0}^{\infty} | c_n^{(h)} | < \infty$; (using Lemma 1). Also $c_n = \delta^h \left(c_n^h\right)$

$\left[c_n^{(h)}\right]$ and so $c_n = o(1)$ as $n \to \infty$ Also, from lemma 4 it follows that (N, p, α) is regular. Hence $\lambda_{n,k} = o(1)$ as $n \to \infty$ and k fixed, which is (2.3). From (2.4) it is evident that $\lambda_{n,n} > 0$, so we consider now $\lambda_{n,k}$ for $k < n$. Applying Abel's transformation h-times to the inner sum in (4.5), we obtain

$$\lambda_{n,k} = \frac{(r * \alpha)_k}{(p * \alpha)_n} \sum_{v=k}^{n} \left[\Delta\left(\frac{\alpha_v}{(q * \alpha)_v} \right)\right]\left[\sum_{u=k}^{v} \left(\Delta_u^h \, p_{n-u} \right) c_{u-k}^{(h)} \right]$$

$$+ \frac{(r * \alpha)_k}{(p * \alpha)_n} \sum_{v=k}^{n} \left[\Delta\left(\frac{\alpha_v}{(q * \alpha)_v} \right)\right]\left[\sum_{l=1}^{h} \left(\Delta_v^{l-1} \, p_{n-v-1} \right) c_{v-k}^{(l)} \right] \qquad \ldots(4.6)$$

$$= \Sigma_1 + \Sigma_2 \text{ (say)}.$$

The condition (iii) of the hypotheses imply $\Delta\left(\frac{\alpha_v}{(q * \alpha)_v}\right) \geqslant 0$ (for $k = 0$) and so by Lemma 2, $\Sigma_k \geqslant 0$. For Σ_2; $\geqslant 0$ by Abel's transformation $(h-l+1)$ times and for $l = 1, 2, 3, \ldots, h$.

$$\sum_{v=k}^{n} \Delta\left[\frac{\alpha_v}{(q * \alpha)_v} \right]\left[\Delta_v^{l-1} \, p_{n-v-1} \right] c_{v-k}^{(l)}$$

$$= \sum_{v=k}^{n} \Delta_v^{h-l+1} \left\{ \Delta\left[\frac{\alpha_v}{(p * \alpha)_v} \right] \left[\Delta_v^{l-1} \, p_{n-v-1} \right] c_{v-k}^{(h+1)} \right.$$

$$= \sum_{j=0}^{h-l+1} \binom{h-l+1}{j} \sum_{v=k}^{n} \left[\Delta^{j+1}\left(\frac{\alpha_v}{(q * \alpha)_v} \right)\right]\left[\Delta_v^{h-j} \, p_{n-v-1+j} \right] c_{v-k}^{(h+1)}.$$

Now $\Delta_v^{h-j} \, p_{n-v-1+j} > 0$; (using (ii) of the hypothesis).

and $c_{v-k}^{(h+1)} \geqslant 0$ (using Lemma 1).

Also $\Delta^{j+1}\left(\frac{\alpha_v}{(q * \alpha)_v} \right) \geqslant 0$ (using condition (iii) of the hypothesis).

Thus we have $\Sigma_2 \geqslant 0$.

Therefore $\lambda_{n,k} \geqslant 0$ $(k \leqslant n)$.

and

$$\sum_{k=0}^{n} |\lambda_{n,k}| = \sum_{k=0}^{n} \lambda_{n,k} = 1.$$

This proves (2.2).

The required inclusion now follows from Theorem 1.

5. Theorems

In this section we establish theorems in which; under appropriate conditions, the class of (N, p, α) (N, q, α)-summable sequences from a sub-class of (N, r, α)-summable sequences.

Theorem 4—Necessary and sufficient conditions that

$$(N, p, \alpha)\,(N, q, \alpha) \subset (N, r, \alpha) \qquad \ldots(5.1)$$

are

$$\sum_{k=0}^{n} |\lambda_{n,k}| = O(1)\ (n \geqslant 0) \qquad \ldots(5.2)$$

and

$$\lambda_{n,k} = o(1) \text{ as } n \to \infty \text{ and } k \text{ fixed} \qquad \ldots(5.3)$$

where

$$\lambda_{n,k} = \begin{cases} \dfrac{(p * \alpha)_k}{(r * \alpha)_n} \displaystyle\sum_{\nu=k}^{n} \dfrac{p_{n-\nu}\,(q * \alpha)_\nu\, c_{\nu-k}}{\alpha_\nu} & (k \leqslant n). \\ 0\ (k > n) \end{cases} \qquad \ldots(5.4)$$

Further if (N, p, α) is regular and either of the conditions hold :

(a) $\dfrac{(q * \alpha)_n}{\alpha_n} \to Q \neq 0$; $c_n = O(1)$ and $\dfrac{1}{\alpha_n} = O(1)$.

or

(b) $p_n \geqslant 0$; $q_n \geqslant 0$; $\alpha_n > 0$; $\dfrac{1}{\alpha_n} = O(1)$ and $c_n = o(1)$.

Then (5.2) alone is the necessary and sufficient condition for the truth of (5.1).

In the particular case when $\alpha_n = 1$ for all n we get the result (Das 1968a, Theorem 4):

Theorem 5 — Let

(i) $q_n \geqslant 0$; $\alpha_n > 0$

(ii) $\{\alpha_n\}$ is non-decreasing

(iii) $\{p_n\} \in M$

(iv) $\left\{ \dfrac{(q * \alpha)_n}{\alpha_n} \right\}$ is non-decreasing

(v) $(n + 1)\, \alpha_n\, \delta\left(\dfrac{(q * \alpha)_n}{\alpha_n}\right) = O\left[(q * \alpha)_n\right]$.

hold. Then (5.1) holds.

In case $\alpha_n = 1$ for all n; we get the result (Das 1968 a, Theorem 5).

Proof of Theorem 4

By definition

$$t^r_{\alpha,n} = \frac{1}{(r * \alpha)_n} \sum_{\nu=0}^{n} r_{n-\nu}\, \alpha_\nu s_\nu.$$

(*equation continued on p.* 1041)

$$= \frac{1}{(r * \alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu} (q * \alpha)_v t^q_{\alpha,\nu} \qquad \ldots(5.5)$$

(using $r_n = (p * q)_n$).

Also

$$t^{p,q}_{\alpha,n} = \frac{1}{(p * \alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu} \alpha_v t^q_{\alpha,\nu} . \qquad \ldots(5.6)$$

By inversion formula we have

$$\alpha_n t^q_{\alpha,n} = \sum_{\nu=0}^{n} c_{n-v} (p* \alpha)_v t^{p,q}_{\alpha,\nu} . \qquad \ldots(5.7)$$

Thus, from (6.1) we get

$$t^r_{\alpha,n} = \frac{1}{(r * \alpha)_n} \sum_{\nu=0}^{n} \frac{p_{n-\nu} (q * \alpha)}{\alpha_\nu} \sum_{k=0}^{\nu} c_{\nu-k} (p* \alpha)_k t^{p,q}_{\alpha,k}$$

$$= \sum_{k=0}^{n} \lambda_{n,k} t^{p,q}_{\alpha,k} ; \text{ where } \lambda_{n,k} \text{ is given by (5.4).}$$

Further, if $s_n = 1$ for all n, then $t^p_{\alpha,n} = 1$ and $t^q_{\alpha,n} = 1$ for all n which will imply $t^{p,q}_{\alpha,n} = 1$ for all n. Therefore $\sum_{k=0}^{n} \lambda_{n,k} = 1$.

Hence (5.2) and (5.3) are the necessary and sufficient conditions for (5.1) to hold. (by Toeplitz's theorem (Hardy 1949, Theorem 2)).

Consider the case when (a) holds. Since $\frac{(q * \alpha)_n}{\alpha_n} \to Q$ and

$$(r * \alpha)_n = (p * q * \alpha)_n = \sum_{\nu=0}^{n} p_{n-\nu} (q* \alpha)_v.$$

From the regularity of (N, p, α) it follows that

$$\frac{(r * \alpha)_n}{(p * \alpha)_n} \sim Q.$$

From (5.4) we have

$$\frac{(r * \alpha)_n}{(p * \alpha)_k (p * \alpha)_n} \lambda_{n,k} = \frac{Q}{(p * \alpha)_n} \sum_{\nu=k}^{n} p_{n-v} c_{v-k}$$

(*equation continued on p.* 1042)

$$+ \frac{1}{(p * \alpha)_n} \sum_{\nu=k}^{n} p_{n-\nu}\, c_{\nu-k} \left[\frac{(q * \alpha)_\nu}{\alpha_\nu} - Q \right]$$

$$= \Sigma_1 + \Sigma_2 \text{ (say)}.$$

Now $\Sigma_1 = 0$ for $n > k$; and for Σ_2; the hypothesis implies that $\frac{c_{\nu-k}}{\alpha_\nu} = O(1)$ and $\left[\frac{(q * \alpha)_\nu}{\alpha_\nu} - Q\right] = o(1)$ as $\nu \to \infty$.

Thus $\Sigma_2 = o(1)$ as $\nu \to \infty$ and for k-fixed, (since (N, p, α) is regular). So we conclude that (5.3) holds.

For case (b) the hypotheses imply $c_{\nu-k} = o(1)$ for $\nu \to \infty$ and k-fixed which further implies $\frac{c_{\nu-k}}{\alpha_\nu} = o(1)$ as $\frac{1}{\alpha_n} = O(1)$. To prove (5.3); it is enough to show that the sequence-to-sequence transformation

$$t_n = \frac{1}{(r * \alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu} (q * \alpha)_\nu S_\nu \text{ is regular; } \left(S_\nu = \frac{c_{\nu-k}}{\alpha_\nu} \right);$$

where the coefficients are all non-negative and their sum is one.

Also $(r * \alpha)_n = (p * q * \alpha)_n \geqslant q_0 (p * \alpha)_n$

and (N, p, α) is regular, so $p_{n-\nu} = o(p * \alpha)_n = o(r * \alpha)_n$ and this gives the conclusion of case (b).

The proof of Theorem 4 is complete.

Corollary 2—Let $q_n \geqslant 0$; $\alpha_n > 0$; $(Q * \alpha)_n \to \infty$ and $\left\{ \frac{(q * \alpha)_n}{\alpha_n} \right\}$ is non-increasing. Then $(\bar{N}, \alpha)$ $(N, q, \alpha) \subset (N, Q, \alpha)$.

This follows from Theorem 4 on putting $p_n = 1$ for all n; In this case $c_0 = 1$; $c_1 = -1$ and $c_n = 0$ for $n > 1$.

For $k < n$.

$$\lambda_{n,k} = \frac{(1 * \alpha)_k}{(1 * q * \alpha)_n} \left[\frac{(q * \alpha)_k}{\alpha_k} - \frac{(q * \alpha)_{k+1}}{\alpha_{k+1}} \right].$$

and for $k = n$ only the last term in the bracket is not there. Therefore the hypotheses imply that $\lambda_{n,k} \geqslant 0$; for $k \leqslant n$, and $(Q * \alpha)_n \to \infty$ implies $\lambda_{n,k} = o(1)$ as $n \to \infty$ and for k fixed.

Thus the inclusion holds.

Corollary 3 — If $q_n \geqslant 0$ the necessary and sufficient condition for $(C, 1)$ $(N, q) \subset (N, Q)$ is given by

$$(n + 1) Q_n \leqslant H (Q_0 + Q_1 + \dots + Q_n).$$

This follows immediately from Theorem 4 when $\alpha_n = 1$ and $p_n = 1$ for all n.

Proof of Theorem 5

The condition (5.3) holds; (using case (b) of Theorem 4). We have only to show now (5.2) holds.

From (5.4) we have

$$\lambda_{n,k} = \frac{(p * \alpha)_k}{(r * \alpha)_n} \sum_{\nu=k}^{n} \frac{p_{n-\nu} (q * \alpha)_\nu c_{\nu-k}}{\alpha_\nu} \quad (k \leqslant n)$$

so that

$$\lambda_{n,n} = \frac{(p * \alpha)_n (q * \alpha)_n}{(r * \alpha)_n \alpha_n} .$$

We first show that $\lambda_{n,n} = O(1)$. This will be so if

$$(p * \alpha)_n (q * \alpha)_n = O[(r * \alpha)_n \alpha_n]. \quad ...(5.8)$$

We write

$$\frac{(q * \alpha)_n}{\alpha_n} = u_n$$

so that we are given that $\{u_n\}$ is positive non-decreasing and that

$$(n+1)\, \delta u_n = O(u_n) \quad ...(5.9)$$

also

$$(r * \alpha)_n = (p * q * \alpha)_n = (p * \alpha u)_n$$

where we write 'αu' for the sequence $\{\alpha_n u_n\}$. Thus the result (5.8), to be proved takes the form

$$(p * \alpha)_n u_n = O[(p * \alpha u)_n]. \quad ...(5.10)$$

By definition

$$(p * \alpha)_n = p_n \alpha_0 + p_{n-1} \alpha_1 + ... + p_0 \alpha_n \quad ...(5.11)$$

since $\{p_n\}$ is non-increasing and $\{\alpha_n\}$ is non-decreasing so the terms on the right of (5.11) are non-decreasing.

Hence, writing $m = \left[\frac{n}{2}\right]$ we have

$$(p * \alpha)_n \leqslant 2 (p_{n-m} \alpha_m + p_{n-m-1} \alpha_{m+1} + \dots + p_0 \alpha_n). \quad ...(5.12)$$

Now condition (5.9) implies that

$$(n+1)(u_n - u_{n-1}) \leqslant H u_n.$$

Thus $u_n \leqslant \dfrac{u_{n-1}}{\left[1 - \dfrac{H}{n+1}\right]}$; provided that the denominator is positive; also for $n = n-1, n-2, \dots, (m+1)$.

$$u_{m+1} \leqslant \frac{u_m}{\left[1 - \dfrac{H}{m+2}\right]}$$; provided that the denominator is positive

Thus

$$u_n \leqslant \frac{u_m}{\left[1 - \dfrac{H}{m+2}\right]\left[1 - \dfrac{H}{m+3}\left[\dots\left[1 - \dfrac{H}{n+1}\right]\right.\right.} \quad ...(5.13)$$

provided that each in the denominator is positive.
Write

$$f(n) = \left[1 - \frac{H}{m+2}\right]\left[1 - \frac{H}{m+3}\right]\cdots\left[1 - \frac{H}{n+1}\right].$$

so

$$\log f(n) = \sum_{\nu=m+1}^{n} \log\left(1 - \frac{H}{\nu+1}\right)$$

$$= \sum_{\nu=m+1}^{n}\left[1 - \frac{H}{\nu+1} + O\left(\frac{1}{\nu^2}\right)\right].$$

$$= -H\log\left(\frac{n}{m}\right) + O\left(\frac{1}{n}\right).$$

Hence

$$f(n) = \left(\frac{n}{m}\right)^{-H}\left[1 + O\left(\frac{1}{n}\right)\right]$$

$$\to \left(\frac{1}{2}\right)^{H} \text{ as } n \to \infty.$$

Thus $u_{2r} = O(u_r)$ and $u_s = O(u_r)$ uniformally in $r \leqslant s \leqslant 2_r$. From (5.12), $(p * \alpha)_n$ (6.8) u_n is less than or equal to a constant times

$$p_{n-m}\alpha_m u_m + p_{n-m-1}\alpha_{m+1}u_{m+1} + \dots + p_0\alpha_n u_n \leqslant (p * \alpha u)_n$$

And (5.10) holds; which implies that (5.8) holds.
We now consider the case $k < n$. By Abel's transformation we have

$$\lambda_{n,k} = \frac{(p * \alpha)_k}{(r * \alpha)_n}\sum_{\nu=k}^{n-1}\Delta\left(\frac{q * \alpha)_\nu}{\alpha_\nu}\right)\sum_{j=k}^{\nu} p_{n-j}\, c_{j-k}$$

$$+ \frac{(q * \alpha)_n (p * \alpha)_k}{\alpha_n (r * \alpha)_n}\sum_{j=k}^{n} p_{n-j}\, c_{j-k}.$$

$$= \lambda^{(1)}_{n,k} + \lambda^{(2)}_{n,k} \text{ (say}$$

But

$$\lambda^{(2)}_{n,k} = 0 \text{ for } n > k,$$

Therefore

$$\lambda_{n,k} = \lambda^{(1)}_{n,k} \text{ for } n > k.$$

Also $\lambda^{(1)}_{n,k} \leqslant 0$; (using Lemma 2 (for $h = 0$) and $\Delta\left(\frac{(q * \alpha)_n}{\alpha_n}\right) \leqslant 0$).

Thus

$$| \lambda_{n,k} | = - \lambda_{n,k}^{(1)} \; (k < n).$$

and we have

$$\begin{aligned}\sum_{k=0}^{n} | \lambda_{n,k} | &= \sum_{k=0}^{n-1} | \lambda_{n,k} | + | \lambda_{n,n} | \\ &= - \sum_{k=0}^{n-1} \lambda_{n,k}^{(1)} + | \lambda_{n,n} | \\ &= - \sum_{k=0}^{n-1} \lambda_{n,k} + | \lambda_{n,n} | \\ &= \sum_{k=0}^{n} \lambda_{n,k} + 2 | \lambda_{n,n} | \\ &= -1 + 2 | \lambda_{n,n} | \\ &= O(1); \text{ (since (5.8) holds).}\end{aligned}$$

This proves the theorem.

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