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ON THE INCLUSION RELATIONS BETWEEN $L^r(\mu)$ AND $L^s(\mu)$

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For a measure space (X, μ), a necessary and sufficient condition is stated under which $L^r(\mu) \subset L^s(\mu)$ for $0 < r < s$. Conditions under which $L^s(\mu) \subset L^r(\mu)$ are also discussed.

Let μ be a positive measure on a nonempty set X. Subramanian (1978) has discussed the conditions under which $L^r(\mu) \subset L^s(\mu)$ for $0 < r < s$. Here we state a simpler necessary and sufficient condition with an easy proof. We also discuss the case under which $L^s(\mu) \subset L^r(\mu)$ for $0 < r < s$.

All the notations are taken from (Rudin 1966).

Let us take $F = \{A \subset X \mid 0 < \mu(A) < \infty\}$. If F is empty then all subsets of X of finite measure will have the measure zero. Therefore for any $r > 0$ and $f \in L^r(\mu)$, the set $N(f) = \{x \mid f(x) \neq 0\}$, which is of σ-finite measure (Halmos 1962, p. 105), will also have the measure zero. Thus for any $r > 0$, $L^r(\mu) = \{0\}$ where $0(x) = 0$ for all x, and therefore in this case we do not have to discuss any inclusion relation. So from now onwards we shall assume that F is non empty.

Theorem 1—For $0 < r < s$, $L^r(\mu) \subset L^s(\mu)$ iff

$\inf\{\mu(A) \mid A \in F\} > 0$.

PROOF : *Sufficiency.*

Suppose $\inf\{\mu(A) \mid A \in F\} = \alpha > 0$.

Let $f \in L^r(\mu)$. Then f must be essentially bounded, otherwise there exists an increasing sequence of integers $\{n_i\}$ such that

$$\mu(A_i) = \mu\{x \mid n_i \leqslant |f(x)| < n_{i+1}\} > 0 \text{ for } i = 1,2,\ldots$$

Hence, by hypothesis, $\mu(A_i) \geqslant \alpha$ for all i. Therefore

$$\int_X |f|^r \, d\mu \geqslant \sum_{i \geqslant 1} n_i^r \, \mu(A_i) = \infty,$$

which contradicts the assumption that $f \in L^r(\mu)$. Hence suppose $|f(x)| \leqslant M < \infty$ for a.e. x in X.

Let $B = \{x \mid |f(x)| \geqslant 1\}$ and $C = \{x \mid |f(x)| < 1\}$.

We see that $\mu(B) \leqslant \int_B |f|^r \, d\mu \leqslant \int_X |f|^r \, d\mu < \infty$.

Now $\int_X |f|^s d\mu = \int_B |f|^s d\mu + \int_C |f|^s d\mu$

$$\leqslant M^s \mu(B) + \int_C |f|^r d\mu$$

$$< \infty.$$

Hence $f \in L^s(\mu)$.

Necessity. Suppose $L^r(\mu) \subset L^s(\mu)$ for $0 < r < s$. If $\inf\{\mu(A) \mid A \in F) = 0$, then we can find a sequence of sets $\{E_i\}$ such that $0 < \mu(E_1) < 1/2$ and $0 < \mu(E_{i+1}) < \mu(E_i)/2^i$ for $i = 1, 2, \ldots$ Take $B_i = E_i - \bigcup_{j>i} E_j$ and $\mu(B_i) = \delta_i$ for all i. Then $\{B_i\}$ is a sequence of pairwise disjoint sets and $0 < \delta_i < 2^{-i}$ for all i.

Define $f(x) = \delta_i^{-1/s}$ if $x \in B_i$

$$= 0 \text{ if } x \notin \bigcup_{i\geqslant 1} B_i.$$

Then f is measurable and

$$\int_X |f|^r d\mu = \Sigma \int_{B_i} |f|^r d\mu = \Sigma\, \delta_i^{-r/s}\, \delta_i < \Sigma\, 2^{-i(1-(r/s))} < \infty.$$

[The sums are taken over all natural values of i.]

But $\int_X |f|^s d\mu = \Sigma \int_{B_i} |f^s| \, d\mu = \Sigma\, \delta_i^{-1}\, \delta_i = \infty$.

Thus $f \in L^r(\mu)$ and $f \notin L^s(\mu)$, which contradicts the hypothesis.

Theorem 2—The following are equivalent:

(i) For $0 < r < s$, $L^s(\mu) \subset L^r(\mu)$;

(ii) $\sup\{\mu(A) \mid A \in F\} < \infty$;

(iii) $\exists\, G \subset X$ with $\mu(G) < \infty$ such that $\mu(A) = \mu(A \cap G)$ for every $A \subset X$ with $\mu(A) < \infty$.

PROOF: (i) $\Rightarrow$ (ii). Suppose (i) holds. If (ii) fails, then we can find a sequence $\{E_i\} \subset F$ such that

$$\mu(E_1) \geqslant 2 \text{ and } \mu(E_i) \geqslant 3\, \mu(\bigcup_{j<i} E_j) \text{ for } i > 1.$$

Take $B_1 = E_1$ and $B_i = E_i - \bigcup_{j<i} E_j$ for $i > 1$.

Then $\{B_i\}$ is a pairwise disjoint sequence and

$$\mu(B_i) \geqslant 2\, \mu(\bigcup_{j<i} E_j) \geqslant 2\mu(\bigcup_{j<i} B_j) = 2 \sum_{j<i} \mu(B_j).$$

Hence, by induction, we can claim that $\mu(B_i) \geqslant 2^i$ for all i.

Take $\mu(B_i) = \delta_i$ for all i and define

$f(x) = \delta_i^{-1/r}$ if $x \in B_i$

$$= 0 \text{ if } x \notin \bigcup_{i\geqslant 1} B_i.$$

Then $\int_X |f|^s \, d\mu = \Sigma \int_{B_i} |f|^s \, d\mu = \Sigma \, \delta_i^{(1-(s/r))} \leqslant \Sigma \, 2^{\, i(1-(s/r))} < \infty$

and $\int_X |f|^r \, d\mu = \Sigma \int_{B_i} |f|^r \, d\mu = \Sigma \, \delta_i^{-1} \, \delta_i = \infty.$

But this contradicts (i).

(ii) $\Rightarrow$ (iii). Suppose $\sup \{\mu(A) \mid A \in F\} = m < \infty$.

Then either, $\exists \, G \in F$ such that $\mu(G) = m$, or, $\exists$ a sequence $\{A_i\} \subset F$ such that $\mu(A_i) > m - (1/i)$ for all i. Taking $G = \bigcup_{i \geqslant 1} A_i$, we get,

$\mu(G) = \lim_{j \to \infty} \mu(\bigcup_{i \leqslant j} A_i) = m$ since

$m - 1/j < \mu(A_j) \leqslant \mu(\bigcup_{i \leqslant j} A_i) \leqslant m$ as $\bigcup_{i \leqslant j} A_i \in F$ for all j.

So, there always exists a set $G \in F$ such that $\mu(G) = m$. Now we claim that this G satisfies the property stated in (iii). On the contrary, suppose $\exists \, B$ such that $\mu(B) < \infty$ and $\mu(B) \neq \mu(B \cap G)$.

Then $\mu(B - G) = \mu(B) - \mu(B \cap G) > 0$.

Hence $\mu(G \cup B) = \mu(G) + \mu(B - G) > m$, which contradicts the initial assumption as $G \cup B \in F$.

(iii) $\Rightarrow$ (i). Suppose (iii) is true. Let $f \in L^s(\mu)$ and $Y = \{x \mid f(x) \neq 0\}$. Since Y is σ—finite (Halmos 1962, p. 105), we have a sequence $\{A_i\}$ of disjoint sets of finite measure such that $Y \subset \bigcup_{i \geqslant 1} A_i$.

Hence,

$$\mu(Y) \leqslant \Sigma \, \mu(A_i) = \Sigma \, \mu(A_i \cap G) \leqslant \mu(G) < \infty.$$

Now, using Holder's inequality, we get

$$\int_X |f|^r \, d\mu = \int_Y |f|^r d\mu \leqslant \left(\int_Y |f|^s \, d\mu\right)^{r/s} (\mu(Y))^{1-(r/s)}.$$

Hence $f \in L^r(\mu)$. Therefore $L^s(\mu) \subset L^r(\mu)$.

Example 1—If μ is a counting measure on N (the set of all natural numbers), then

$$\inf \{\mu(A) \mid A \subset N, 0 < \mu(A) < \infty\} = 1.$$

Hence, by Theorem 1, $l^r = L^r(\mu) \subset L^s(\mu) = l^s$ for $0 < r < s$.

Example 2—For any Lebesgue measurable subset A of the real line, define

$\mu(A) = m(A \cap [0, 1])$ if $A - [0, 1]$ is countable

$= \infty$, otherwise

where m is the Lebesgue measure on the real line. Then, for $0 < r < s$, $L^s(\mu) \subset L^r(\mu)$ by Theorem 2.

References

Halmos, P. R. (1962). Measure Theory. D. Van Nostrand Company, Inc., New York.

Rudin, W. (1966). Real and Complex Analysis. McGraw-Hill Book Co. Inc., New York.

Subramanian, B. (1978). On the inclusion $L^p(\mu) \subset L^q(\mu)$. *Am. Math. Monthly*, **85**, 479–81.