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AN EXPANSION THEOREM FOR A SECOND-ORDER SINGULAR MATRIX DIFFERENTIAL EQUATION

BIKAN BHAGAT AND GUMA SWESI

University of Al-Fateh, Post Box 13045, *Tripoli, Libya*

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We consider the second-order matrix differential equation

$$(N - \lambda)\phi = 0, \quad 0 \leqslant x < \infty,$$

where N is a second-order matrix differential operator and ϕ a vector having two components. The general expansion formula

$$f(x) = \frac{1}{\pi} \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(x, \mu)\, d F_r(\mu), \; 0 \leqslant x < \infty,$$

has been established.

§ 1. Titchmarsh (1962) in his book has developed the direct convergence theory for the real second-order differential equation

$$\psi^{(2)}(x) + (\lambda - q(x))\,\psi(x) = 0, \;\; 0 \leqslant x < \infty, \qquad \ldots(1.1)$$

applying his analytical methods to the singular case in Chapters II and III. In his paper (1961) he has extended the theory to the matrix differential equation of the first-order

$$(L - \lambda)\phi = 0, \;\; 0 \leqslant x < \infty, \qquad \ldots(1.2)$$

where L denotes the matrix operator

$$L = \begin{pmatrix} p(x) & q(x) + \dfrac{d}{dx} \\ q(x) - \dfrac{d}{dx} & r(x) \end{pmatrix},$$

$p(x)$, $q(x)$ and $r(x)$ being given real valued functions of x, $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$ a vector having two components. [For special cases see Conte and Sangren (1954) and Roos and Sangren (1963).]

In the present paper we are concerned with the problem of extending the results of Titchmarsh (1961) to the second-order matrix differential equation

$$(N - \lambda)\phi = 0, \;\; 0 \leqslant x < \infty, \qquad \ldots(1.3)$$

where N denotes the second-order matrix differential operator

$$N = \begin{pmatrix} -\dfrac{d}{dx}\left(p_0(x)\dfrac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & -\dfrac{d}{dx}\left(q_0(x)\dfrac{d}{dx}\right) + q_1(x) \end{pmatrix}, \qquad \ldots(1.4)$$

ϕ is a vector having two components $u \equiv u(x)$ and $v \equiv v(x)$ represented as a column matrix $\begin{pmatrix} u \\ v \end{pmatrix}$, and λ is a parameter, real or complex.

The coefficients $p_0(x)$, $q_0(x)$, $p_1(x)$, $q_1(x)$ and $r(x)$ satisfy the following conditions

(i) $p_0(x)$, $q_0(x)$ are real valued and absolutely continuous on all compact subintervals of $[0, \infty)$;

(ii) $p_0(x), q_0(x) > 0$ for all $x \in [0, \infty)$;

(iii) $p_1(x)$, $q_1(x)$ and $r(x)$ are all real valued and continuous on $[0, \infty)$.

The bilinear concomitant $[g\,h](x) = [g(x)\,h(x)]$ of any two differentiable vectors $g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$ and $h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}$ is defined as

$$[gh](x) = p_0(g_1'\bar{h}_1 - g_1\bar{h}_1') + q_0(g_2'\bar{h}_2 - g_2\bar{h}_2'), \quad \ldots(1.5)$$

and Green's formula is given by

$$\int_a^b (g^T N\bar{h} - \bar{h}^T Ng)\,dx = [gh](b) - [gh](a), \quad \ldots(1.6)$$

where $0 \leqslant a < b < \infty$.

The boundary conditions to be satisfied by any solution ϕ of (1.3) at $x = 0$, are taken in the form

$$[\phi\,\phi_j](0) = 0, \quad j = 1, 2, \quad \ldots(1.7)$$

where

$$\phi_j \equiv \phi_j(0/x, \lambda) = \begin{pmatrix} u_j(0/x, \lambda) \\ v_j(0/x, \lambda) \end{pmatrix} \quad (j = 1, 2)$$

are the boundary condition vectors such that

$$[\phi_1\,\phi_2] = 0. \quad \ldots(1.8)$$

§ 2. To obtain an expansion theorem for eqn. (1.3) and the boundary conditions (1.7) we define a matrix differential operator T on the Hilbert space $L^2[0, \infty)$. The domain of such an operator T may well determine the set of vector functions for which the expansion is valid.

Define the set $\mathscr{D}(T)$ of vectors f as follows :

$f \in \mathscr{D}(T) \subset L^2$ if

(i) $f \in L^2[0, \infty)$,

(ii) f' is absolutely continuous on $[0, b]$ for all $b > 0$,

(iii) $Nf \in L^2[0, \infty)$,

(iv) $[f\,\phi_j](0) = 0$ for $j = 1, 2$,

(v) $\lim_{b \to \infty} [f\,\dot{\psi}_r](b) = 0$ for $r = 1, 2$ and all complex λ.

Let the operator T be defined by

$$T : \mathscr{D}(T) \to L^2, \quad Tf = Nf \text{ for all } f \in \mathscr{D}(T).$$

The main theorem to be proved is the following expansion theorem for the second-order matrix differential equation (1.3).

Theorem — Let the vector $f(x) \in \mathscr{D}(T)$,

$$\chi_r(x, \mu) = \sum_{s=1}^{2} \int_0^{\mu} \phi_s(x, \mu)\, dk_{rs}(\mu) \qquad (r = 1, 2),$$

where $$k_{rs}(\mu) = \lim_{\nu \to 0} \int_0^{\mu} -\operatorname{im}\,[m_{rs}(\mu + i\nu)]\, d\mu,$$

and $$F_r(\mu) = \int_0^{\infty} \chi_r^T(x, \mu) f(x)\, dx.$$

Then

$$f(x) = \frac{1}{\pi} \sum_{r=1}^{2} \int_{-\infty}^{\infty} \phi_r(x, \mu)\, dF_r(\mu) \qquad (0 \leqslant x < \infty).$$

We shall use the results and notations of Bhagat (1969 a, b). One can also see Chakravarty (1965, 1968).

§ 3. As in the analysis of Chapters II and III of Titchmarsh (1962) and that of Chaudhury and Everitt (1969) and Titchmarsh (1961), we need some of the properties of the boundary-value problem on the finite interval $[0, b]$ determined by the system of differential equations (1.3), the boundary conditions (1.7) at $x = 0$ with condition (1.8) and similar conditions at $x = b$. It has been shown by Bhagat (1969a) that such a boundary-value problem is self-adjoint and determines real eigenvalues $\{\lambda_{n,b} : n \geqslant 1\}$ such that

$$-\infty < \lambda_{1,b} \leqslant \lambda_{2,b} \leqslant \ldots \leqslant \lambda_{n,b} \leqslant \ldots$$

and a corresponding set of eigenvectors $\{\psi_{n,b}(x) : n \geqslant 1\}$, all real valued on $[0, b]$, which form an orthonormal set i. e.

$$\int_0^b \psi_{n,b}^T(x)\, \psi_{m,b}(x)\, dx = \delta_{nm}.$$

These eigenvectors are complete in the space $L^2[0, b]$ and satisfy the Parseval relation

$$\sum_{n=1}^{\infty} c_{n,b}^2 = \int_0^b f^T(x) f(x)\, dx \qquad \ldots(3.1)$$

for all $f(x) \in L^2[0, b]$, where $c_{n,b} = \int_0^b \psi_{n,b}^T f\, dx,\ n \geqslant 1.$ ...(3.2)

We now indicate the lines of proof of some of the properties of $\Phi(x, \lambda, f)$ for the interval $[0, \infty)$ defined in § 9 of Bhagat (1969b).

Lemma 3.1 — For all $f \in \mathscr{D}(T)$, $\lambda \Phi(x, \lambda, f)$

$$= f(x) + \Phi(x, \lambda, Nf) \ (0 \leqslant x < \infty). \qquad \ldots(3.3)$$

Using Green's formula (1.6) and condition (iv) of § 2, we have

$$\lambda \int_0^x \phi_j^T(y, \lambda) f(y)\, dy = \int_0^x \phi_j^T Nf\, dy + [\phi_j f](x), \qquad (j = 1, 2) \qquad \ldots(3.4)$$

and using condition (v) of § 2, we have

$$\lambda \int_x^\infty \psi_r^T(y, \lambda) f(y)\, dy = \int_x^\infty \psi_r^T Nf(y)\, dy - [\psi_r f](x), \qquad (r = 1, 2). \qquad \ldots(3.5)$$

Hence substituting, from (3.4) and (3.5) in the explicit formula for $\Phi(x, \lambda, f)$, we obtain

$$\lambda \Phi(x, \lambda, f) = \psi_1 \int_0^x \phi_1^T Nf\, dy + \psi_2 \int_0^x \phi_2^T Nf\, dy + \phi_1 \int_x^\infty \psi_1^T Nf\, dy$$

$$+ \phi_2 \int_x^\infty \psi_2^T Nf\, dy + \psi_1 [\phi_1 f](x) + \psi_2 [\phi_2 f](x)$$

$$- \phi_1 [\psi_1 f](x) - \phi_2 [\psi_2 f](x).$$

Now using Lemma (8.1) of Bhagat (1969b), it follows that

$$\psi_1 [\phi_1 f](x) + \psi_2 [\phi_2 f](x) - \phi_1 [\psi_1 f_1](x) - \phi_2 [\psi_2 f](x) = f(x).$$

Hence the result follows.

Lemma 3.2—If $f(x) \in L^2[0, \infty)$, x fixed and im $\lambda = \nu \neq 0$, $|\lambda| > A > 0$ then

$$\Phi_1(x, \lambda, f), \Phi_2(x, \lambda, f) = O(|\lambda|^{1/4} / |\nu|). \qquad \ldots(3.6)$$

The proof follows exactly using the method of § 2 of Bhagat (1975).

Lemma 3.3 — For all real valued vector functions $f \in \mathscr{D}(T)$, there exists a parameter $K(x)$ depending on x only such that

$$\int_{\mu_1}^{\mu_2} |\operatorname{im} \Phi_r(x, f)|\, d\mu < K(x), \quad r = 1, 2,$$

where $\lambda = \mu + i\nu$, $\mu_1 \leqslant \mu \leqslant \mu_2$ and $\nu > 0$.

The proof follows exactly following the methods of § 12.7 of Titchmarsh (1958) and § 9 of Chaudhury and Everitt (1969).

§ 4. It is known (Bhagat (1969 b), § 4) that the only singularities of $l_{rs}(b; \lambda)$ for each fixed b, are simple poles on the real axis. Let $\lambda_{n,b}$ be a simple pole of $l_{rs}(b; \lambda)$ with residue $R_{rs}(b; n)$. Then from (4.6) of Bhagat (1969b), as $\nu \to 0$ (μ fixed)

$$\int_0^\infty i\nu\, \psi_r^T(b; x, \lambda_{n,b} + i\nu)\,(-i\nu\, \psi_r(b; x, \lambda_{n,b} - i\nu))\, dx < \infty, \quad r = 1, 2. \qquad \ldots(4.1)$$

and

$$\int_0^b (i\nu\, \psi_r\,(b;\,x,\,\lambda_{n,b}+i\nu)-U_r\,(b;\,x,\,\lambda_{n,b}))^T\,(-i\nu\,\psi_r\,(b;\,x,\,\lambda_{n,b}-i\nu)-U_r\,(b;x,\lambda_{n,b}))dx$$

tends to zero, where

$$U_r\,(b;\,x,\,\lambda_{n,b}) = \sum_{s=1}^{2} R_{rs}\,(b;\,n)\,\phi_s\,(x,\,\lambda_{n,b}). \qquad \ldots(4.2)$$

Hence $i\nu\,\psi_r\,(b;\,x,\,\lambda_{n,b}+i\nu)\,(r=1,2)$ converge in the mean to $U_r\,(b;\,x,\,\lambda_{n,b})$ uniformly with respect to b as $\nu \to 0$.

Also we have (from Bhagat (1969b, § 4)

$$\int_0^b \psi_r^T\,(b;\,x,\,\lambda_1)\,\psi_s\,(b;\,x,\,\lambda_2)\,dx = \frac{l_{sr}\,(b;\,\lambda_2 - l_{rs}\,(b;\,\lambda_1)}{\lambda_1 - \lambda_2}\,. \qquad \ldots(4.3)$$

Putting $\lambda_2 = \lambda_{n,b} + i\nu$ in (4.3) and multiplying by $i\nu$ and then making $\nu \to 0$, we get

$$\int_0^b \psi_r^T\,(b;\,x,\,\lambda_1)\,U_s\,(b;\,x,\,\lambda_{n,b})\,dx = \frac{R_{rs}}{\lambda_1 - \lambda_{n,b}}\,,\quad \lambda_1 \neq \lambda_{n,b}. \qquad ..(4.4)$$

Putting $\lambda_1 = \lambda_{m,b} + i\nu\;(\lambda_{m,b} \neq \lambda_{n,b})$ and making $\nu \to 0$, we have

$$\int_0^b U_r^T\,(b;\,x,\,\lambda_{m,b})\,U_s\,(b;\,x,\,\lambda_{n,b})\,dx = 0. \qquad \ldots(4.5)$$

If we put $\lambda_1 = \lambda_{n,b} + i\,\nu$, then making $\nu \to 0$, we get

$$\int_0^b U_r^T(b;\,x,\,\lambda_{n,b})\,U_s\,(b;\,x,\,\lambda_{n,b})\,dx = R_{rs}\,(b;\,n). \qquad \ldots)4.6)$$

If $\lambda_{n,b}$ is a simple zero of $D\,(\lambda)$, then from § 7 of Bhagat (1969a), by considering the limits of $i\,\nu\,\psi_r\,(b;\,x,\,\lambda_{n,b} + i\,\nu)\;(r = 1,\,2)$ as $\nu \to 0$ we get

$$R_{11}\,(b;\,n)\,R_{22}\,(b;\,n) = R_{12}^2(b;\,n); \qquad \ldots(4.7)$$

and the normalized eigenvectors $\psi_{n,b}\,(x)$ corresponding to the eigenvalue $\lambda_{n,b}$ can be put in the form

$$\psi_{n,b}^{(r)}\,(x) = R_{rr}^{-\frac{1}{2}}\,(b;\,n)\,U_r\,(b;\,x,\,\lambda_{n,b})\,(r = 1,\,2).$$

By similar arguments it can be shown that if $\lambda_{n,b}$ is a double zero of $D\,(\lambda)$, then

$$R_{11}\,(b;\,n)\,R_{22}\,(b;\,n) - R_{12}^2\,(b;\,n) = \frac{1}{I_{11}\,I_{22}\,I_{12}^2} > 0 \qquad \ldots(4.8)$$

where $I_{jk} = \int_0^b \phi_j^T\,\phi_k\,dx$, $\phi_j\,(j = 1,\,2)$ and $\phi_k\,(k = 3,\,4)$ being the boundary condition vectors at $x = 0$ and $x=b$ respectively. The two orthonormal eigenvectors $\psi_{n,b}^{(r)}(x)$, $r = 1, 2$ corresponding to the eigenvalue $\lambda_{n,b}$ are given by

$$\psi_{n,b}^{(1)}(x) = R_{11}^{-\frac{1}{2}} U_1(b; x, \lambda_{n,b}) \qquad \ldots(4.9)$$

$$\psi_{n,b}^{(2)}(x) = \frac{R_{12} U_1(b; x, \lambda_{n,b}) - R_{11} U_2(b; x, \lambda_{n,b})}{R_{11}^{\frac{1}{2}} (R_{11} R_{22} - R_{12}^2)^{\frac{1}{2}}} .$$

In this case a suitable linear combination of $\psi_{n,b}^{(1)}$ and $\psi_{n,b}^{(2)}$ is taken as a normalized eigenvector corresponding to the eigenvalue $\lambda_{n,b}$.

§ 5. In this section we prove some of the results corresponding to § 3.2, § 3.3 and 3.4 of Titchmarsh (1962).

Lemma 5.1 — For $-\infty < \mu_1 < \mu_2 < \infty$, $0 < \nu \leqslant 1$,

$$\left| \int_{\mu_1}^{\mu_2} - \text{im}\, [m_{rs}(\mu + i\nu)]\, d\mu \right| \leqslant B(\mu_1, \mu_2), \quad 1 \leqslant r, s \leqslant 2. \qquad \ldots(5.1)$$

If $\psi_{n,b}(x)$ is the eigenvector corresponding to the eigenvalue $\lambda_{n,b}$, we have, from (4.4), (4.7), (4.8) and (4.9),

$$\int_0^b \psi_r^T(b; x, \lambda)\, \psi_{n,b}(x)\, dx = \frac{C_{rr}(b; n)}{\lambda - \lambda_{n,b}}, \quad (\nu \neq 0) \qquad \ldots(5.2)$$

where $C_{rr}(b; n)$ are functions of $R_{rs}(b; n)$ $(1 \leqslant r, s \leqslant 2)$.

Now exactly following Titchmarsh (1962), p. 54 we can prove that

$$\int_{\mu_1}^{\mu_2} - \text{im}\, l_{rr}(b; \lambda)\, d\mu \;| \leqslant B(\mu_1, \mu_2) < \infty,$$

for $r = 1, 2$ and $0 < \nu \leqslant 1$.

For real vectors F and G

$$(F_1 G_1 + F_2 G_2)^2 \leqslant (F_1^2 + F_2^2)(G_1^2 + G_2^2)$$

whence $\int_a^b F^T G\, dx \leqslant \{ \int_a^b F^T F\, dx \int_a^b G^T G\, dx \}^{\frac{1}{2}}$ the Schwarz inequality. ...(5.3)

For this and (4.7) of Bhagat (1969 b)

$$| - \text{im}\, l_{rs}(b; \lambda) | \leqslant \{ (- \text{im}\, l_{rr}(b; \lambda)) \, (- \text{im}\, l_{ss}(b; \lambda)) \}^{1/2}.$$

Hence

$$\left| \int_{\mu_1}^{\mu_2} - \text{im}\, l_{rs}(b; \mu + i\nu)\, d\mu \right| \leqslant B(\mu_1, \mu_2) < \infty. \qquad \ldots(5.4)$$

Since $l_{rs}(b; \lambda)$ converge boundedly to $m_{rs}(\lambda)$ as $b \to \infty$, the result (5.1) follows.

Lemma 5.2 — The symmetric matrix

$$K(\lambda) = (k_{rs}(\lambda)) = (\lim_{\nu \to 0} \int_0^{\lambda} -\text{im}\, [m_{rs}(\mu + i\nu)]\, d\mu)$$

exists for all real λ; $K(\lambda)$ is a function of bounded variation, and

$$K(\lambda) = \tfrac{1}{2}\{K(\lambda+0) + K(\lambda-0)\},$$

and the symmetric matrix $K(\lambda)$ is non-decreasing for increasing λ. Also

$$\lim_{\nu \to +0} \int_0^{\mu} -\operatorname{im}[\psi_r(x, \mu + i\nu)]\, d\mu = \sum_{s=1}^{2} \int_0^{\mu} \phi_s(x, \mu)\, dk_{rs}(\mu).$$

Putting $\lambda_2 = \lambda$, $\lambda_1 = i$ in (4.3), we get, using inequality (5.3)

$$| l_{rs}(b; \lambda) | \leqslant | l_{rs}(b; i) | + | i - \lambda | \{ \int_0^b | \psi_r(b; x, \lambda) |^2 dx \int_0^b | \psi_s(b; x, i) |^2 dx \}^{\frac{1}{2}}.$$

Hence if $\lambda = \mu + i\nu$, $\mu_1 \leqslant \mu \leqslant \mu_2$, $0 < \nu \leqslant 1$, we have

$$| l_{rs}(b; \mu + i\nu) | \leqslant \alpha + \beta \left\{ \frac{-\operatorname{im} l_{rr}(b; \mu + i\nu)}{\nu} \right\}^{\frac{1}{2}}$$

where

$$\alpha = \alpha(\mu_1, \mu_2) = | l_{rs}(b; i) | = O(1) \text{ as } b \to \infty,$$

$$\beta = \beta(\mu_1, \mu_2) = | \lambda - i | \ \{ | -\operatorname{im} l_{ss}(b; i) | \}^{1/2} = O(1) \text{ as } b \to \infty.$$

Since $l_{rs}(b; \lambda) \to m_{rs}^{(2)}$ as $b \to \infty$ (through a sequence of values if necessary), we get

$$| m_{rs}(\mu + i\nu) | \leqslant \alpha + \beta \left\{ \frac{-\operatorname{im}[m_{rr}(\mu + i\nu)]}{\nu} \right\}^{\frac{1}{2}}.$$

Thus

$$\int_{\mu_1}^{\mu_2} | m_{rs}(\mu + i\nu) | \, d\mu \leqslant \beta_1(\mu_1, \mu_2) + \frac{\rho_2(\mu_1, \mu_2)}{\nu^{1/2}} .$$

Hence

$$\int_0^1 d\nu \int_{\mu_1}^{\mu_2} | m_{rs}(\mu + i\nu) | \, d\mu < \infty.$$

So from Fubini's theorem on multiple integrals we se that

$$\int_0^1 | m_{rs}(\mu + i\nu) | \, d\nu \quad (1 \leqslant r, s \leqslant 2)$$

are finite for all μ in $\mu_1 \leqslant \mu \leqslant \mu_2$.

Now from the general theorem on the boundary values of analytic functions (Titchmarsh (1958), § 22.23) it follows that the limit $K(\lambda)$ exists and is of bounded variation.

From (5.4), exactly following Chaudhury and Everitt (1969) it can be shown that the matrix $K(\lambda)$ is non-decreasing.

Now

$$\int_0^{\mu} -\operatorname{im}[\psi_r(x, \mu + i\nu)]\, d\mu = \int_0^{\mu} -\operatorname{im}[\theta_r(x, \mu + i\nu)]\, d\mu$$

(*equation continued on p.* 1056)

$$+\int_0^{\mu}\{-\sum_{s=1}^{2}\text{re}\,[m_{rs}(\mu+i\nu)]\,\text{im}\,[\phi_s(x,+i\nu)\}\,d\mu$$

$$+\int_0^{\mu}\{-\sum_{s=1}^{2}\text{im}\,[m_{rs}(\mu+i\nu)]\;\text{re}\,[\phi_s(x,\mu+i\nu)\}\,d\mu.$$

Since the vectors $\theta_r(x,\lambda)$ and $\phi_r(x,\lambda)$ are real for real λ, im $\theta_r(x,\mu+i\nu)$ and im $\phi_r(x,\mu+i\nu)$ are $0(\nu)$, uniformly with respect to μ over a finite interval. Hence the first two terms tend to zero with ν. Let

$$k_{rs}(\mu+i\nu)=\int_0^{\mu}-\text{im}\,[m_{rs}(\mu+i\nu)]\,d\mu.$$

Then the third term tends to

$$\Big[\sum_{s=1}^{2}k_{rs}(\mu)\,\phi_s(x,\mu)\Big]_0^{\mu}-\int_0^{\mu}\sum_{s=1}^{2}k_{rs}(\mu)\,\frac{\partial\,\phi_s(x,\mu)}{\partial\,\mu}\,d\mu\text{ as }\nu\to 0,$$

$=\int_0^{\mu}\sum_{s=1}^{2}\phi_s(x,\mu)\,d\,k_{rs}(\mu)$, since $k_{rs}(\mu)$ are of bounded variation and ϕ_s, $\frac{\partial\phi_s}{\partial\mu}$ are continuous.

Lemma 5.3—Let $\chi_r(x,\mu)=\sum_{s=1}^{2}\int_0^{\mu}\phi_s(x,\mu)\,d\,k_{rs}(\mu)\ (r=1,2)$ ...(5.5)

$0\leqslant x<\infty$ and $-\infty<\mu<\infty$. Then the vectors $\chi_r(x,\mu)\in L^2[0,\infty)$.
From (5.2)

$$\int_0^{b}\psi_{n,b}(x)\,d\,x\int_0^{\mu}\text{im}\,\psi_r(b;x,\mu+i\nu)\,d\mu$$

$$=C_{rr}(b;n)\int_0^{\mu}\frac{-\nu\,d\,\mu}{(\mu-\lambda_{n,b})^2+\nu^2}=O\left(\frac{C_{rr}(b;n)}{(\lambda^2_{n,b}+1)}\right).$$

Hence the Parseval formula gives

$$\int_0^{b}\Big[\Big\{\int_0^{\mu}\text{im}\,\psi_r^T(b;x,\mu+i\nu)\,d\mu\Big\}\Big\{\int_0^{\mu}\text{im}\,\psi_r(b;x,\mu+i\nu)\,d\,\mu\Big\}\Big]\,dx=0$$

$$\Big\{\sum_{n=1}^{\infty}\frac{C^2_{rr}(b;n)}{\lambda^2_{n,b}+1}\Big\}=O\ (1).$$

Making $b\to\infty$, (possibly through a sequence of values) and $\nu\to 0$, it follows that

$$\int_0^{\infty}\chi_r^T(x,\mu)\,\chi_r(x,\mu)\,d\,x<K<\infty,\ K\text{ being an absolute constant.}$$

This proves Lemma 5.3.

From (4.7) of Bhagat (1969b), we have

$$\int_{\mu_1}^{\mu_2} d\mu \int_0^b \psi_r^T (b; x, \mu + i\nu)\, \psi_s (b; x, \mu - i\nu)\, dx = \int_{\mu_1}^{\mu_2} - \frac{\operatorname{im} l_{rs} (b; \mu + i\nu)}{\nu}\, d\mu.$$

Hence as in Lemma 5.1 it follows that, as $\nu \to 0$.

$$\int_{\mu_1}^{\mu_2} d\mu \int_0^\infty \psi_r^T (x, \mu + i\nu)\, \psi_r (x, \mu + i\nu)\, dx = O\left(\frac{1}{\nu}\right), \text{ for fixed } \mu_1 \text{ and } \mu_2.$$

For $r = 1, 2$, define

$$F_r(\mu) = \int_0^\infty \chi_r^T (x, \mu) f(x)\, dx,\ \mu \in (-\infty, \infty).$$

Then from Lemma 5.3, $F_r(\cdot)$ is finite for all real μ and for all real vectors $f \in \mathscr{D}(T)$.

Then exactly following § 3.6 of Titchmarsh (1962), it follows that the variation of $F_r(\mu)$ over the compact interval $[-R, R]$ is finite and is uniformly bounded for all $R > 0$;

(variation of $F_r(\mu)$ over $[-R, R]$) $\leqslant K < \infty$, $(r = 1, 2)$,

for all $R > 0$ where K is an absolute constant.

§ 6. Let $f(x)$ be a real valued vector belonging to $\mathscr{D}(T)$. Then following § § 2.15 and 3.6 of Titchmarsh (1962), we have

$$f(x) = \lim_{R \to \infty} -\frac{1}{\pi i} \int_{-R+i\delta}^{R+i\delta} \Phi(x, \lambda)\, d\lambda, \qquad \ldots(6.1)$$

where $\delta > 0$, and then following Roos and Sangren (1963), § 8, it can be proved that (6.1) can be replaced by

$$f(x) = \lim_{R \to \infty} -\frac{1}{\pi} \int_{-R+i\delta}^{R+i\delta} \operatorname{im} \Phi(x, \lambda)\, d\lambda. \qquad \ldots(6.2)$$

It is then the question of the behaviour of this integral as $\delta \to 0$.

$\phi_j(x, \lambda)$, $\theta_j(x, \lambda)$ $(j = 1, 2)$ are analytic functions of λ and real for real λ, it follows that each of

$$\operatorname{im}(u_j),\ \operatorname{im}(v_j),\ \operatorname{im}(x_j),\ \operatorname{im}(y_j) \qquad \ldots(6.3)$$

is equal to $O(\delta)$ as $\delta \to 0$. Therefore, for x, y in the fixed interval

$$\operatorname{im} [\psi_{r1}(x, \lambda)\, u_j(y, \lambda) - u_j(x, \lambda)\, \psi_{r1}(y, \lambda)] = O(\delta)\ (1 \leqslant r, j \leqslant 2) \qquad \ldots(6.4)$$

and

$$\operatorname{im} [\psi_{11}(x, \lambda)\, v_1(y, \lambda) - u_1(x, \lambda)\, \psi_{12}(y, \lambda) + \psi_{21}(x,\lambda) v_2(y, \lambda) - u_2(x, \lambda)\, \psi_{22}(y, \lambda)] = O(\delta). \qquad \ldots(6.5)$$

Now using explicitt expression for $\Phi(x, \lambda, f)$ and following Titchmarsh (1962), § 3.4, it follows from (6.3), (6.4) and (6.5) that as $\delta \to 0$, for fixed R,

$$-\frac{1}{\pi}\int_{-R+i\delta}^{R+i\delta} \operatorname{im} \Phi_1(x, \lambda)\, d\lambda \text{ is equal to the Stieltjes integral}$$

$$\frac{1}{\pi}\sum_{r=1}^{2}\int_{-R}^{R} u_r(x, \mu)\, dF_r(\mu).$$

Hence

$$f_1(x) = \frac{1}{\pi}\sum_{r=1}^{2} \lim_{R\to\infty}\int_{-R}^{R} u_r(x, \mu)\; d\,F_r(\mu).$$

Similarly,

$$f_2(x) = \frac{1}{\pi}\sum_{r=1}^{2} \lim_{R\to\infty}\int_{-R}^{R} v_r(x, \mu)\, d\,F_r(\mu).$$

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