Indian J. pure appl. Math., 13(9) : 1059–1065, September 1982

ON THE LOGARITHMIC *H*-PROXIMATE ORDER OF ANALYTIC FUNCTIONS REPRESENTED BY DIRICHLET SERIES II

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(*Received* 14 *September* 1981)

To compare the growth of functions represented by Dirichlet series, which are analytic in the half-plane are of same logarithmic order and are of infinite logarithmic type, Awasthi and Dixit (1980) introduced the concept of logarithmic *H*-proximate order. In the present paper we derive relations which depict how the growth of maximum term is closely connected with that of central index and logarithmic *H*-proximate order.

§ 1. Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \qquad \dots (1.1)$$

where $0 < \lambda_n < \lambda_{n+1} \to \infty$, $s = \sigma + it$ (σ, t being real variables) $\{a_n\}_1^{\infty}$ is sequence of complex numbers and

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D < \infty . \qquad \dots (1.2)$$

If the series given by (1.1) converges absolutely in the half-plane Re $s < \alpha$ ($-\infty < \alpha < \infty$), then it is known Mandelbrojt (1944, p. 166) that the series (1.1) represents an analytic function in Re $s < \alpha$ and since (1.2) holds we have

$$\alpha = -\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} . \qquad \dots (1.3)$$

Set,

$$M(\sigma) = \underset{-\infty < t < \infty}{\text{l.u.b.}} |f(\sigma + it)|$$

$$m(\sigma) = \max_{n \geqslant 1} |a_n| \exp(\sigma\lambda_n)$$

and

$$N(\sigma) = \max\{n : m(\sigma) = |a_n| \exp(\sigma\lambda_n)\}.$$

$M(\sigma)$, $m(\sigma)$ and $N(\sigma)$ are called respectively, the maximum modulus maximum term and the rank of maximum term or the central index of $f(s)$ for Re $s = \sigma$.

Let D_α^* denote the class of all functions $f(s)$ of zero order which are analytic in the half-plane Re $s < \alpha$ and are defined by (1.1). We shall sav th at $f(s) \in A_\alpha^* \subset D_\alpha^*$ if if and only if $m(\sigma)/\{1-\exp(\sigma-\alpha)\}^{-\epsilon} \to \infty$, as $\sigma \to \alpha$ for some $\epsilon > 0$. The logarithmic order ρ^* and lower logarithmic order λ^* of $f(s) \in A_\alpha^*$ are defined (Awasthi and Dixit 1979) as

$$\lim_{\sigma\to\alpha} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log\log\{1-\exp(\sigma-\alpha)\}^{-1}} = \begin{matrix}\rho^*\\ \lambda^*\end{matrix}, \quad (1 \leqslant \lambda^* \leqslant \rho^* < \infty). \qquad \ldots(1.4)$$

By confining to the notion of logarithmic order only, it is not possible to compare precisely the growth of functions having same finite logarithmic order. For this purpose the concept of the logarithmic type has been introduced (Awasthi and Dixit 1980a). Thus

Definition—The function $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \in A_\alpha^*$, having logarithmic order ρ^* $(1 < \rho^* < \infty)$ is said to be of logarithmic type T^* and lower logarithmic type t^* if

$$\lim_{\sigma\to\alpha} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*}} = \begin{matrix}T^*\\ t^*\end{matrix}. \qquad \ldots(1.5)$$

But it is clear that these concepts are not sufficient to compare the growth of those functions of the class A_α^* which are of same logarithmic order and of infinite logarithmic type. For this purpose we introduced (Awasthi and Dixit 1980b) the concept of logarithmic H-proximate order $\rho^*(\sigma)$ and the concept of generalized logarithmic type $\overline{T}^*$ as below:

Definition 1—A real-valued function $\rho^*(\sigma)$ defined on $(-\infty, \alpha)$ is called a logarithmic H-proximate order if it satisfies the following conditions:

$\rho^*(\sigma)$ is a positive, continuous and piecewise differentiable function for all σ such that $-\infty < \sigma_0 < \sigma < \alpha$; ...(1.6)

$$\lim_{\sigma\to\alpha} \rho^*(\sigma) = \rho^* \ (1 < \rho^* < \infty); \qquad \ldots(1.7)$$

$$\lim_{\sigma\to\alpha} [\rho'_*(\sigma)\{1-\exp(\sigma-\alpha)\}\log\{1-\exp(\sigma-\alpha)\}^{-1} \times \{\log\log(1-\exp(\sigma-\alpha))^{-1}\}] = 0, \qquad \ldots(1.8)$$

where $\rho^{*\prime}(\sigma)$ is either the right or left derivative of $\rho^*(\sigma)$ where these are different.

The generalized logarithmic type $\overline{T}^*$ and lower generalized logarithmic type $\overline{t}^*$ of $f(s) \in A_\alpha^*$ with respect to a given logarithmic H-proximate order $\rho^*(\sigma)$ are defined (Awasthi and Dixit 1980b) as

$$\lim_{\sigma\to\alpha} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)}} = \begin{matrix}\overline{T}^*\\ \overline{t}^*\end{matrix} \qquad \ldots(1.9)$$

$$(0 \leqslant \overline{t}^* \leqslant \overline{T}^* \leqslant \infty).$$

Definition 2—A logarithmic *H*-proximate order $\rho^*(\sigma)$ is called a logarithmic *H*-proximate order of $f(s)$ belonging to A_α^* if $\overline{T}^*$ is nonzero finite.

Given a logarithmic *H*-proximate order $\rho^*(\sigma)$ and a function $f(s) \in A_\alpha^*$, we define

$$\begin{matrix} Q \\ q \end{matrix} = \lim_{\sigma\to\alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda_{N(\sigma)}\,[1 - \exp(\sigma-\alpha)]}{[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)-1}\exp(\sigma-\alpha)} \quad \ldots(1.10)$$

$$\begin{matrix} P \\ p \end{matrix} = \lim_{\sigma\to\alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda_{N(\sigma)}\,[1 - \exp(\sigma-\alpha)]}{[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)}\exp(\sigma-\alpha)} \quad \ldots(1.11)$$

and

$$\begin{matrix} B \\ b \end{matrix} = \lim_{\sigma\to\alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda_{N(\sigma)}\,[\log\{1-\exp(\sigma-\alpha)\}^{-1}]\{1-\exp(\sigma-\alpha)\}}{\{\exp(\sigma-\alpha)\}\log m(\sigma)}. \quad \ldots(1.12)$$

In this paper we obtain some relations between a logarithmic *H*-proximate order of the function $f(s)$ in A_α^* and its maximum term and central index. The results given below in the form of theorems depict how the growth of maximum term of $f(s)$ is closely connected with that of the central index and logarithmic *H*-proximate order. We shall assume, throughout this section, that $f(s)$ in A_α^* satisfies and that all the constants involved are non-zero finite and q, Q, p, P and b, B are respectively given by (1.10), (1.11) and (1.12).

§2. In the following theorem we obtain relations for logarithmic *H*-proximate order $\rho^*(\sigma)$ interms of the constants Q, q, B, b. For this, first we prove the following lemma.

Lemma 1—Let $\rho^*(\sigma)$ be a logarithmic *H*-proximate order, then for $\xi < \rho^* + 1$ and for $\sigma_0 < \beta < \sigma < \alpha$, we have

$$\int_\beta^\sigma \frac{[\log\{1-\exp(t-\alpha)\}^{-1}]^{\rho^*(t)+\xi}\exp(t-\alpha)}{1-\exp(t-\alpha)}\,dt$$

$$= \frac{1}{\rho^*+1-\xi}\,[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)+1-\xi}$$

$$+\; O\,[\log\{1-\exp(\sigma-\alpha)^{-1}]^{\rho^*(\sigma)+1-\xi}\,. \quad \ldots(2.1)$$

PROOF :

$$\text{L.H.S} = \int_\beta^\sigma \frac{\log\{1-\exp(t-\alpha)\}^{-1}]^{\rho^*-\xi}\,[\log\{1-\exp(t-\alpha)\}^{-1}]^{\rho^*(t)-\rho^*}\exp(t-\alpha)}{1-\exp(t-\alpha)}\,dt$$

$$= \left[\frac{1}{\rho^*+1-\xi}\,[\log\{1-\exp(t-\alpha)\}^{-1}]^{\rho^*(t)+1-\xi}\right]_\beta^\sigma$$

(*equation continued on p.* 1062)

$$- \frac{1}{\rho^*+1-\xi} \int_{\beta}^{\sigma} \Big[\log (1-e^{t-\alpha})^{-1}\}^{\rho^*(t)-\xi} \{\log (1-e^{t-\alpha})^{-1}\}$$

$$\times \rho^{*'} (t) \log \log (1-e^{t-\alpha})^{-1} (1-e^{t-\alpha}) \frac{+ (\rho^* (t)-\rho^*) \exp (t-\alpha)}{1-\exp (t-\alpha)} \Big] dt.$$

Now from the definition of logarithmic H-proximate order, we have asymptotically

$$| \rho^*(t)-\rho^* | < \epsilon/2$$

and

$$| \{1-\exp (t-\alpha)\} \log \{1-\exp (t-\alpha)^{-1} \rho' (t) \log \log \{1-\exp (t-\alpha)\}^{-1} | < \epsilon/2.$$

Using this, we have (2.1).

Following Lemma is due to Krishnandan (1973).

Lemma 2—If $f (s) = \sum_{n=1}^{\infty} a_n \exp (s\lambda_n)$ belongs to the class D_α where D_α denotes the class of all Dirichlet series of the form (1.1) which satisfy (1.2), are not exponential polynomials and the sum function of whose represents an analytic function in the half plane Re $s < \alpha$, then

$$\log m(\sigma) = \log m(\sigma_1) + \int_{\sigma_1}^{\sigma} \lambda_{N(u)} \, du, \quad - \infty < \sigma_1 < \sigma < \alpha. \qquad \ldots(2.2)$$

Now we have

Theorem 1—Let $f (s)$ belonging to A_α^* be of logarithmic order ρ^*, lower logarithmic order λ^* $(1 < \lambda^* \leqslant \rho^* < \infty)$, and of logarithmic H-proximate order $\rho^* (\sigma) \rightarrow \rho^*$, as $\sigma \rightarrow \alpha$. If the constants Q, q and B, b be given by (1.10) and (1.12) respectively, then

$$b \leqslant \lambda^* \leqslant \rho^* \leqslant B, \qquad \ldots(2.3)$$

$$\frac{q}{Q} \leqslant \frac{\rho^*}{B} \leqslant \frac{\rho^*}{b} \leqslant \frac{Q}{q} . \qquad \ldots(2.4)$$

PROOF : By (1.12), we have for every ϵ such that $b > \epsilon > 0$ and for all σ such that $- \infty < \sigma_0 (\epsilon) = \sigma_0 < \sigma < \alpha$

$$(b-\epsilon) < \frac{\{1-\exp (\sigma-\alpha)\}\lambda_{N(\sigma)} [\log \{1-\exp (\sigma-\alpha)\}^{-1}]}{\exp (\sigma-\alpha) \log m(\sigma)} < B+\epsilon.$$

Also, by (2.2) we get $\frac{m' (\sigma)}{m(\sigma)} = \lambda_{N(\sigma)}$ for almost all values of σ for $- \infty < \sigma < \alpha$.

Therefore

$$\frac{(b-\epsilon)\exp(\sigma-\alpha)}{\{1-\exp(\sigma-\alpha)\}\log\{1-\exp(\sigma-\alpha)\}^{-1}} < \frac{m'(\sigma)}{m(\sigma)\log m(\sigma)}$$

$$< \frac{(B+\epsilon)\exp(\sigma-\alpha)}{\{1-\exp(\sigma-\alpha)\}\log\{1-\exp(\sigma-\alpha)\}^{-1}}.$$

Now integrating the above inequalities from σ_0 to σ, we have

$$(b-\epsilon)\log\log\{1-\exp(\sigma-\alpha)\}^{-1}-O(1) < \log\log m(\sigma)$$

$$< (B+\epsilon)\log\log\{1-\exp(\sigma-\alpha)\}^{-1}-O(1).$$

(2.3) follows on dividing by $\log\log\{1-\exp(\sigma-\alpha)\}^{-1}$ and proceeding to limits as $\sigma\to\alpha$.

Again from (1.10), we have for every ϵ such that $q>\epsilon>0$, and for all σ such that $-\infty<\sigma_0(\epsilon)=\sigma_0<\sigma<\alpha$,

$$q-\epsilon < \frac{\{1-\exp(\sigma-\alpha)\}\lambda_{N(\sigma)}}{\exp(\sigma-\alpha)[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)-1}} < Q+\epsilon. \qquad \ldots(2.5)$$

This together with (2.2) gives

$$\log m(\sigma) < \log m(\sigma_0)+(Q+\epsilon)\int_{\sigma_0}^{\sigma}\frac{[\log\{1-\exp(t-\alpha)\}^{-1}]^{\rho^*(t)-1}\exp(t-\alpha)}{[1-\exp(t-\alpha)[}\,dt.$$

Now applying Lemma 1, with $\xi=1$, we get

$$\log m(\sigma) < \log m(\sigma_0) + \frac{Q+\epsilon}{\rho^*}\left[\log\left\{1-\exp(\sigma-\alpha)\right\}^{-1}\right]^{\rho^*\sigma}$$

$$+\ O\left[\log\left\{1-\exp(\sigma-\alpha)\right\}^{-1}\right]^{\rho^*(\sigma)}.$$

Therefore for all σ such that $\alpha>\sigma>\sigma_0>-\infty$,

$$\frac{\exp(\sigma-\alpha)\log m(\sigma)}{\{1-\exp(\sigma-\alpha)\}\lambda_{N(\sigma)}\log\{1-\exp(\sigma-\alpha)\}^{-1}}$$

$$< O(1) + \frac{(Q+\epsilon)[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)-1}\exp(\sigma-\alpha)}{\rho^*[1-\exp(\sigma-\alpha)]\lambda_{N(\sigma)}}.$$

Passing to limits we have

$$\frac{1}{b}\leqslant\frac{Q}{\rho^*}\,\frac{1}{q}. \qquad \ldots(2.6)$$

By similar arguments the inequality on the L.H.S. of (2.5) gives

$$\frac{1}{B}\geqslant\frac{q}{Q\rho^*}. \qquad \ldots(2.7)$$

(2.6) with (2.7) gives (2.4).

Corollary — If $Q=q$, then $B=b=\rho^*=\lambda^*$ i.e. $f(s)$ is of regular logarithmic growth.

In the following theorem we obtain relations for generalized logarithmic type $\overline{T}^*$ and lower generalized logarithmic type $\overline{t}^*$ in terms of the constants p, P, q, Q.

Theorem 2—*Let* $f(s)$, belonging to A^*_α, be of logarithmic order ρ^* $(1<\rho^*<\infty)$ and let it be of generalized logarithmic type $\overline{T}^*$ and lower generalized logarithmic type $\overline{t}^*$ with respect to its logarithmic H-proximate order $\rho^{*(\sigma)}$. If Q and q, P and p are given by (1.10) and (1.11); then we have

$$\frac{q}{\rho^*}+\frac{P}{K}\leqslant \overline{T}^*\leqslant \frac{Q}{\rho^*}+\frac{P}{K}\ ;\ \frac{q}{\rho^*}+\frac{p}{K}\leqslant \overline{t}^*\leqslant \frac{Q}{\rho^*}+\frac{p}{K}\ . \qquad \ldots(2.8)$$

PROOF : By (1.10), we have for every ϵ such that $q>\epsilon>0$ and for all σ such that $\alpha>\sigma>\sigma_0-\sigma_0(\epsilon)$

$$\{1-\exp(\sigma-\alpha)\}\lambda_{N(\sigma)}>(q-\epsilon)\exp(\sigma-\alpha)[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)-4}$$

Further, for $1<K<\infty$ ...(2.9)

$$\int_{\sigma}^{\sigma+\left[\left(\frac{1}{K}\right)\left\{1-\exp\sigma-\alpha\right\}\right]}\lambda_{N(t)}\,dt\geqslant\{1-\exp(\sigma-\alpha)\}\lambda_{N(\sigma)}/K \qquad \ldots(2.10)$$

therefore (2.2) and (2.9) give,

$$\log m\left\{\sigma+\frac{1}{K}\left(1-\exp(\sigma-\alpha)\right)\right\}$$

$$=\log m(\sigma_0)+\int_{\sigma_0}^{\sigma}\lambda_{N(t)}dt+\int_{\sigma}^{\sigma+\left(\frac{1}{K}\right)\left\{1-\exp\sigma-\alpha\right\}}\lambda_{N(t)}\,dt$$

$$>\log m(\sigma_0)+(q-\epsilon)\int_{\sigma}^{\sigma}\frac{\exp(t-\alpha)[\log\{1-\exp(t-\alpha)\}^{-1}]^{\rho^*(t)-1}}{1-\exp(t-\alpha)}\,dt$$

$$+\frac{\lambda_{N(\sigma)}\{1-\exp(\sigma-\alpha)\}}{K}. \qquad \ldots(2.11)$$

Now applying Lemma 1 with $\xi=1$, we get

$$\log m\left[\sigma+\frac{1}{K}\left\{1-\exp(\sigma-\alpha)\right\}\right]$$

$$>\log m(\sigma_0)+\frac{(q-\epsilon)}{\rho^*}\left[\log\left\{1-\exp\left(\sigma-\alpha\right)\right\}^{-1}\right]^{\rho^*(\sigma)}$$

$$+\frac{\lambda_{N(\sigma)}\{1-\exp(\sigma-\alpha)\}}{K}+O[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)}.$$

Dividing the above inequality by $[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)}$ and proceeding to limits and using (1.10) and (1.11) we get

$$\overline{T}^*\geqslant\frac{q}{\rho^*}+\frac{P}{K} \qquad \ldots(2.12)$$

$$\bar{t}^* \geqslant \frac{q}{\rho^*} + \frac{p}{K}. \qquad \ldots(2.13)$$

In a similar way, using the definition of Q we easily get

$$\bar{T}^* \leqslant \frac{Q}{\rho^*} + \frac{p}{K} \qquad \ldots(2.14)$$

$$\bar{t}^* \leqslant \frac{Q}{\rho^*} + \frac{p}{K} \qquad \ldots(2.15)$$

From (2.12), (2.13) and (2.14), (2.15) we have (2.8).

In particular, since (2.1 3) and (2.14) hold for all K such that $1 < K < \infty$, letting $K \to \infty$ in these inequalities we have the following theorem.

Theorem 3— Let $f(s)$ belonging to A^*_α be of logarithmic order ρ^* $(1 < \rho^* < \infty)$ and let it be of generalized logarithmic type $\bar{T}^*$ and lower generalized logarithmic type $\bar{t}^*$ with respect to its logarithmic H-proximate order $\rho^*(\sigma)$ and let the logarithmic H-proximate order $\rho^*(\sigma) \to \rho^*$ as $\sigma \to \alpha$. If Q and q are given by (1.10), then we have

$$q \leqslant \rho^* \bar{t}^* \leqslant \rho^* \bar{T} \leqslant Q. \qquad \ldots(2.16)$$

Corollary — If $\lambda_{N(\sigma)} \dfrac{\rho^* \bar{T}^* \exp(\sigma-\alpha)\,[\log\{1-\exp(\sigma-\alpha)\}^{-1}\}^{\rho^*(\sigma)-1}}{\{1\text{-}\exp(\sigma-\alpha)\}}$

as $\sigma \to \alpha$, then $f(s)$ is of perfectly regular logarithmic growth with respect to $\rho^*(\sigma)$ and $\bar{T}^* = Q/\rho^*$.

PROOF : Since $\lambda_{N(\sigma)} \sim \dfrac{\rho^* \bar{T}^* \exp(\sigma-\alpha)\,[\log\{1-\exp(\sigma-\alpha)\}^{-1}]^{\rho^*(\sigma)-1}}{1-\exp(\sigma-\alpha)}$

therefore from (1.10) $Q=q=\rho^* \bar{T}^*$, and so it follows immediately from (2.16) that $\bar{T}^* = \bar{t}^* = Q/\rho^*$. Hence $f(s)$ is of perfectly regular logarithmic growth.

ACKNOWLEDGEMENT

The author wishes to accord his warm thanks to Dr K. N. Awasthi, D.A.V. College, Kanpur, for his guidance in the preparation of this note. He is also grateful to Dr O. P. Juneja for his kind encouragement.

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