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INSTABILITY OF HYDROMAGNETIC FLOW OF STRATIFIED FLUID BETWEEN TWO ROTATING CYLINDERS

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The stability of hydromagnetic flow of radially stratified fluid between two rotating coaxial cylinders is investigated. The relaxation of Boussinesq approximation reduces the rate of increase of instability in the case of axisymmetric disturbances. The tendency of non-axisymmetric disturbances to propagate against the basic rotation is relatively insensitive when the Boussinesq approximation is relaxed but the conditions for the amplification of non-axisymmetric disturbances have greater dependence on density gradient. Further a quadrant theorem on the 'slow' waves characteristic of a rapidly rotating fluid is derived which shows that the relaxation of Boussinesq approximation makes the system more unstable.

1. INTRODUCTION

Nature has provided mechanisms for bringing about changes in temperature that give rise to variations in density. Many meteorological phenomena are manifestations of stratified flow. In particular the hydromagnetic wave like instabilities in a rotating stratified fluid may help to study how far such waves within the earth's liquid core may be responsible for the slow westward drift with time of the geomagnetic field. This hydromagnetic stability of rotating stratified fluid has been investigated by several authors Rudraiah (1964), Braginsky (1967), Acheson (1972 and 1973), (see also Acheson and Hide 1973). At this stage, it is of interest to understand the full effects of the variations of density on the hydromagnetic stability of rotating fluid.

The study of Acheson (1972) pertaining to hydromagnetic stability of rotating homogeneous fluid shows that the system can always be rendered stable to axisymmetric disturbances by sufficient rapid rotation and unless the magnetic field everywhere decreases with radius, the system may be unstable to non-axisymmetric disturbances even when the rotation speed exceeds a typical Alfvén speed by many orders of magnitude. The study also shows that slow hydromagnetic waves may be generated by the spatial variation and all such unstable waves so generated propagate against the basic rotation i.e. 'westward' when the field is azimuthal.

The effects of density distribution subject to Boussinesq approximation along with the magnetic field have been considered by Acheson (1973) and as in the homogeneous case, the system is thoroughly stable to axisymmetric disturbances if the

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fluid rotates 'rapidly' and the magnetic field is moderate. It has been shown that in the case of non-axisymmetric disturbances, all amplifying waves propagate westward. The general conditions necessary for the amplification of such modes along with a quadrant theorem showing the region in which 'slow' waves are unstable have been obtained.

This study of Acheson is confined to Boussinesq fluids. We relax the Boussinesq approximation and carry out the stability analysis to obtain more quantitative informations about the system.

2. Mathematical Formulation

The equations governing the motion of an inviscid, perfectly conducting, incompressible stratified fluid are (see Chandrasekar 1961)

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) = -\nabla p + \frac{1}{\mu}\mathbf{B}.\nabla\mathbf{B} + \rho\mathbf{g} \qquad ...(1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \qquad ...(2)$$

$$\nabla.\mathbf{u} = 0 \qquad ...(3)$$

$$\nabla.\mathbf{B} = 0 \qquad ...(4)$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u}.\nabla\rho = 0. \qquad ...(5)$$

Here ρ denotes the fluid density, u the Eulerian velocity vector, t the time, μ magnetic permeability, B magnetic field, g acceleration due to gravity and p the total pressure which includes both the fluid pressure p_F and the magnetic pressure $1/2\mu\, B^2$.

We choose a set of cylindrical polar coordinates (r, θ, z) relative to which the basic equilibrium state

$$\mathbf{u}_0 = [0, U_\theta(r), U_z(r)],\ \mathbf{B}_0 = [0, B_\theta(r), B_z(r)],\ \rho = \rho_0(r) \qquad ...(6)$$

is an exact solution of (1) to (5) provided that the gravitational body force $\mathbf{g} = [g(r), 0, 0]$ per unit mass is purely radial. This also provides the relation

$$\frac{dp_0}{dr} = \rho_0\left(g + \frac{U_\theta^2}{r}\right) - \frac{B_\theta^2}{\mu r}. \qquad ...(7)$$

We here study the stability of the above flow between two coaxial cylinders of radii r_1 and r_2 and of infinite length. Thus if we slightly disturb the system, then we will have

$$\mathbf{u} = [\hat{u}_r, U_\theta + \hat{u}_\theta, U_z + \hat{u}_z],\ \mathbf{B} = [\hat{b}_r, B_\theta + \hat{b}_\theta, B_z + \hat{b}_z]$$

$$p = p_0 + \hat{p},\ \rho = \rho_0 + \hat{\rho} \qquad ...(8)$$

where the quantities with cap are the deviations from the basic state and are assumed to be small to permit linearization. The equations of motion that govern the disturbed motion then reduce to a set of linear partial differential equations whose coefficients are pure functions of r only (independent of θ. z, t) and permit solutions of the form

$$\hat{\varphi} = \kappa \left[\varphi(r) \exp i (m\theta + nz - \sigma t) \right]$$

where m, n and σ (may be complex) are constants. Then we get

$$-iu_r\omega - \frac{2U_\theta u_\theta}{r} = -\frac{p'}{\rho_0} + \frac{ib_r}{\mu\rho_0}\left(m\frac{B_\theta}{r} + nB_z\right) - \frac{2B_\theta b_\theta}{\mu\rho_0 r} + \frac{\rho}{\rho_0}\left(g + \frac{U_\theta^2}{r}\right) \quad \ldots(9)$$

$$-iu_\theta\omega + u_r U_\theta' + \frac{U_\theta u_r}{r} = -\frac{imp}{\rho_0 r} + \frac{1}{\mu\rho_0}\left[b_r B_\theta' + ib_\theta\left(m\frac{B_\theta}{r} + nB_z\right) + \frac{B_\theta b_r}{r}\right] \quad \ldots(10)$$

$$-iu_z\omega + u_r U_z' = -\frac{inp}{\rho_0} + \frac{1}{\mu\rho_0}\left[b_r B_z' + ib_z\left(m\frac{B_\theta}{r} + nB_z\right)\right] \quad \ldots(11)$$

$$-ib_r\omega = iu_r\left(\frac{mB_\theta}{r} + nB_z\right) \quad \ldots(12)$$

$$-ib_\theta\omega = b_r\left(U_\theta' - \frac{u_\theta}{r}\right) - u_r\left(B_\theta' - \frac{B_\theta}{r}\right) + iu_\theta\left(\frac{mB_\theta}{r} + nB_z\right) \quad \ldots(13)$$

$$-ib_z\omega = b_r U_z' - u_r B_z' + iu_z\left(\frac{mB_\theta}{r} + nB_z\right) \quad \ldots(14)$$

$$u_r' + \frac{u_r}{r} + i\left(\frac{mu_\theta}{r} + nu_z\right) = 0 \quad \ldots(15)$$

$$b_r' + \frac{b_r}{r} + i\left(m\frac{b_\theta}{r} + nb_z\right) = 0 \quad \ldots(16)$$

$$-i\rho\omega + u_r\rho_0' = 0 \quad \ldots(17)$$

where primes denote differentiation with respect to r and

$$\omega(r) \equiv \sigma - m\frac{U_\theta}{r} - nU_z. \quad \ldots(18)$$

Eliminating all perturbation variables in favour of the radial velocity component u_r and defining the local Alfvén speeds

$$V_\theta(r) \equiv \frac{B_\theta(r)}{(\mu\rho_0)^{1/2}}\ ;\ V_z(r) \equiv \frac{B_z(r)}{(\mu\rho_0)^{1/2}} \quad \ldots(19)$$

the Brunt-Väisälä frequency

$$N(r) \equiv \left[\frac{\rho_0'}{\rho_0}\left(g + \frac{U_\theta^2}{r} - \frac{V_\theta^2}{r}\right)\right]^{1/2} \quad \ldots(20)$$

and the functions

$$F(r) \equiv \left(\frac{mV_\theta}{r} + nV_z\right)^2 - \omega^2 \quad \ldots(21)$$

$$Q(r) \equiv 2\left[\frac{U_\theta}{r}\omega + \frac{V_\theta}{r}\left(\frac{mV_\theta}{r} + nV_z\right)\right] \quad \ldots(22)$$

$$\eta(r) \equiv \frac{iu_r}{\omega} \qquad \ldots(23)$$

we obtain

$$F\eta'' + \left[F' + \frac{\rho_0'}{\rho_0} F + \frac{F}{r}\left(\frac{3m^2 + n^2r^2}{m^2 + n^2r^2}\right)\right]\eta' + H\eta = 0 \qquad \ldots(24)$$

where
$$H(r) = \frac{2nU_z'}{r}(\sigma - nU_z) + \frac{\rho_0'}{\rho_0}\left(\frac{F}{r} - \frac{mQ}{r}\right) + \frac{2n^2V_zV_z'}{r} + \frac{Q^2n^2}{F}$$
$$+ n^2r\left[\left(\frac{V_\theta^2}{r^2}\right)' - \left(\frac{U_\theta^2}{r^2}\right)'\right] - N^2\left(\frac{m}{r^2} + n^2\right) + \frac{2mQ}{(r^2 + m^2n^{-2})}$$
$$- \frac{F}{(r^2 + m^2n^{-2})}\left[1 + r^2n^2 + 2m^2 + \frac{m^2}{n^2r^2}(m^2 - 1)\right]. \qquad \ldots(25)$$

To find the nature of the characteristic values of eqn. (24), we use the boundary conditions that the normal component of velocity is zero at the walls of the cylinders i.e. $\eta(r_1) = \eta(r_2) = 0$

We take $U_z = 0$, so that the last term in (18) and the first term in (25) vanish immediately.

Further we take $\psi = e^{-\beta r/2}\eta$, where $\beta = -\rho_0'/\rho_0$ is always positive. Then (24) and (25) transform into

$$F\psi'' + \left[F' + \frac{F}{r}\left(\frac{3m^2 + n^2r^2}{m^2 + n^2r^2}\right)\right]\psi' + H\psi = 0 \qquad \ldots(26)$$

where
$$H(r) = -\frac{F\beta^2}{4} + \frac{F'\beta}{2} + \frac{F\beta}{2r}\left(\frac{m^2 - n^2r^2}{m^2 + n^2r^2}\right) + \frac{\beta mQ}{r} + \frac{2n^2V_zV_z'}{r}$$
$$+ n^2r\left[\left(\frac{V_\theta^2}{r^2}\right)' - \left(\frac{U_\theta^2}{r}\right)'\right] - N^2\left(\frac{m^2}{r^2} + n^2\right) + \frac{Q^2n^2}{F}$$
$$+ \frac{2mQ}{(r^2 + m^2n^{-2})} - \frac{F}{(r^2 + m^2n^{-2})}\left[r^2n^2 + 1 + 2m^2 + \frac{m^2}{n^2r^2}(m^2 - 1)\right] \qquad \ldots(27)$$

with boundary conditions $\psi(r_1) = \psi(r_2) = 0$.

3. Stability Analysis for Axisymmetric Disturbances

We consider the case in which the magnetic field is purely azimuthal ie $V_z = 0$. By taking $m = 0$, (26) reduces to

$$\psi'' + \frac{1}{r}\psi' - \left[\frac{n^2L}{\omega^2} + \frac{\beta^2}{4} + \frac{\beta}{2r} + n^2 + \frac{1}{r^2}\right]\psi = 0 \qquad \ldots(28)$$

where
$$L(r) \equiv r\frac{d}{dr}\left(\frac{V_\theta^2}{r^2}\right) - N^2 - \frac{1}{r^3}\frac{d}{dr}(rU_\theta)^2 \qquad \ldots(29)$$

and $\omega = \sigma$ (constant).

Introducing the transformation $\xi = r^{1/2}\psi$, we find

$$\xi'' - \left(\frac{n^2L}{\omega^2} + \frac{\beta^2}{4} + \frac{\beta}{2r} + n^2 + \frac{3}{4r^2}\right)\xi = 0 \qquad \ldots(30)$$

with boundary conditions $\xi(r_1) = \xi(r_2) = 0$. Multiplying (30) by the complex conjugate of ξ, integrating between r_1 and r_2 and using the boundary conditions, we obtain,

$$\omega^2 \int_{r_1}^{r_2} \left[|\xi'|^2 + \left(\frac{\beta^2}{4} + \frac{\beta}{2r} + n^2 + \frac{3}{4r^2} \right) |\xi|^2 \right] dr = - \int_{r_1}^{r_2} Ln^2 |\xi|^2 \, dr. \qquad \ldots(31)$$

This shows that ω^2 is real and hence the disturbances either oscillate about the equilibrium position without amplifying ($\omega^2 > 0$) or grow a periodically ($\omega^2 < 0$). So, using (31) we conclude that the system is stable if $L \leqslant 0$ and unstable if $L > 0$. This result is similar to that obtained by Acheson (1973) but the relaxation of Boussinesq approximation has modified the Brunt-Väisälä frequency and using (29) we conclude that the relaxation of Boussinesq approximation reduces the rate of increase of instability.

4. Non-axisymmetric Amplifying Waves

We take $\omega = \omega_R + i\omega_I$ so that $\omega_R = \sigma_R - mU_\theta r$ is a function of r and $\omega_I = \sigma_I$ is a constant. Equation (26) may be rewritten as

$$\left(\frac{r^3 F \psi'}{r^2 + m^2 n^{-2}} \right)' + \left(\frac{r^3 H \psi}{r^2 + m^2 n^{-2}} \right) = 0. \qquad \ldots(32)$$

Multiplying by the complex conjugate of ψ and integrating between $r = r_1$ and $r = r_2$ we obtain

$$\int_{r_1}^{r_2} \frac{r^3}{(r^2 + m^2 n^{-2})} \left[H(r) |\psi|^2 - F(r) |\psi'|^2 \right] dr = 0. \qquad \ldots(33)$$

By considering the imaginary part of (33), we get

$$\omega_I \int_{r_1}^{r_2} \frac{r^3}{(r^2 + m^2 n^{-2})} \left\{ \omega_R |\psi'|^2 + \left[\frac{\beta m U_\theta}{r^2} - \omega_R' \frac{\beta}{2} + \omega_R \frac{\beta^2}{4} \right.\right.$$

$$\left.\left. + \, \omega_R \frac{\beta}{2r} \left(\frac{r^2 - m^2 n^{-2}}{r^2 + m^2 n^{-2}} \right) + S_1(r) + \frac{2mU_\theta}{r\,(r^2 + m^2 n^{-2})} + \frac{4n^2 S_2(r)}{|F|^2} \right] |\psi|^2 \right\} dr = 0 \qquad \ldots(34)$$

where $$S_1(r) \equiv \frac{\omega_R}{(r^2 + m^2 n^{-2})} \left[1 + 2m^2 + n^2 r^2 + \frac{m^2}{n^2 r^2} (m^2 - 1) \right] \qquad \ldots(35)$$

$$S_2(r) \equiv \omega_R \left(\frac{U_\theta^2 + V_\theta^2}{r^2} \right) \left(\frac{mV_\theta}{r} + nV_z \right)^2 + \frac{U_\theta V_\theta}{r^2} \left(\frac{mV_\theta}{r} + nV_z \right)$$

$$\times \left[\left(\frac{mV_\theta}{r} + nV_z \right)^2 + \omega_R^2 + \omega_I^2 \right]. \qquad \ldots(36)$$

We consider the case in which the uniform angular velocity $\Omega(r) = U_\bullet r^{-1}$ of the fluid is in the same sense (positive, say) throughout $r_1 \leqslant r \leqslant r_2$. If the magnetic field is purely azimuthal i.e. $V_z = 0$, then (34) becomes

$$\omega_I \int_{r_1}^{r_2} \frac{r^3 \omega_R}{(r^2 + m^2 n^{-2})} \left[|\psi'|^2 + P(r) |\psi|^2 \right] dr = 0 \qquad \ldots(37)$$

where
$$P(r) \equiv \frac{\beta m}{2r\omega_R}\left(U_\theta' + \frac{U_\bullet}{r} \right) + \frac{\beta^2}{4} + \frac{\beta}{2r}\left(\frac{r^2 - m^2 n^{-2}}{r^2 + m^2 n^{-2}} \right)$$
$$+ \frac{1}{(r^2 + m^2 n^{-2})}\left[1 + r^2 n^2 + 2m^2 + \frac{m^2}{n^2 r^2}(m^2 - 1) + \frac{2m\Omega}{\omega_R} \right]$$
$$+ \frac{4n^2}{|F|^2} \frac{m^2 V_\theta^2}{r^2}\left[\frac{V_\theta^2}{r^2} + \Omega^2 + \frac{\Omega}{m\omega_R}\left(\frac{m^2 V_\theta}{r^2} + \omega_R^2 + \omega_I^2 \right) \right]. \qquad \ldots(38)$$

Suppose that a non-axisymmetric disturbance is unstable (i.e. $\omega_I > 0$) but that it does not propagate (i.e. $\omega_R = 0$). Then

$$\omega_I \Omega m \int_{r_1}^{r_2} \frac{r^3 |\psi|^2}{(r^2 + m^2 n^{-2})}\left[\frac{\beta}{2r} + \frac{1}{(r^2 + m^2 n^{-2})} + \frac{2n^2 V_\theta^2}{|F|^2 r^2}\left(\frac{m^2 V_\theta^2}{r^2} + \omega_I^2 \right) \right] dr = 0. \qquad \ldots(39)$$

Using $F = \frac{m^2 V_\theta^2}{r^2} + \omega_I^2$, (39) simplifies to

$$\omega_I \Omega m \int_{r_1}^{r_2} \frac{r^3 |\psi|^2}{(r^2 + m^2 n^{-2})}\left[\frac{\beta}{2r} + \frac{1}{(r^2 + m^2 n^{-2})} + \frac{2n^2 V_\theta^2}{(m^2 V_\theta^2 + r^2 \omega_I^2)} \right] dr = 0 \qquad \ldots(40)$$

which is a contradiction, for the integrand is positive throughout the interval and the integral therefore cannot vanish.

Hence the non-axisymmetric unstable modes must propagate.

Further, from (38) we conclude that $\omega_R/m < 0$, for otherwise a similar contradiction arises from the fact that $P(r)$ would then be positive throughout the interval. So, all unstable modes generated in a uniformly rotating stratified fluid by non-uniformities of the basic azimuthal magnetic field and the radial density stratification in the fluid must propagate against the basic rotation i.e. 'westward'. Evidently this result confirms the result obtained with Boussinesq approximation by Acheson (1973).

5. Conditions Necessary for the Amplification of Non-Axisymmetric Disturbances

We consider the case of uniform rotation and azimuthal magnetic field, so that ω_R is constant. Dividing (33) by ω^2 and equating the imaginary part of the resulting left-hand side to zero, we obtain

$$\omega_R\, \omega_I \int_{r_1}^{r_2} \frac{r^3}{(r^2 + m^2 n^{-2})} \left\{ \frac{m^2 V_\theta^2}{r^2} \left|\frac{\psi'}{\omega}\right|^2 + \left(\frac{4 n^2 \Omega^2 |\omega|^2}{|F|^2} \right.\right.$$

$$\left.\left. + S_3 + S_4 + S_5 \right) |\psi|^2 \right\} dr = 0 \qquad \ldots(41)$$

where
$$S_3 (r) \equiv - \frac{n^2}{|\omega|^2} \left[r \left(\frac{V_\theta^2}{r^2} \right)' - N^2 \left(1 + \frac{m^2}{r^2 n^2} \right) + \frac{\beta m^2}{2 n^2} \left(\frac{V_\theta^2}{r^2} \right)' \right] \ldots(42)$$

$$S_4 (r) \equiv - \frac{2 \Omega m}{\omega_R} \left[\frac{\beta}{2 r} + \frac{2 n^2 V_\theta^2}{|F|^2 r^2} \left(\frac{m^2 V_\theta^2}{r^2} - 3 \omega_R^2 + \omega_I^2 \right) + \frac{1}{(r^2 + m^2 n^{-2})} \right] \ldots(43)$$

$$S_5 (r) \equiv \frac{m^2 V_\theta^2}{|\omega|^2 r^2} \left[\frac{\beta^2}{4} - \frac{\beta}{2 r} \left(\frac{5 m^2 + 3n^2 r^2}{m^2 + n^2 r^2} \right) + \frac{S_1 (r)}{\omega_R} - \frac{4}{r^2 + m^2 n^{-2}} \right.$$

$$\left. - \frac{4 n^2}{m^2} \left\{ 1 + \frac{r^2 |\omega|^4}{m^2 V_\theta^2} \left(\frac{m^2 V_\theta^2}{r^2} - 2 \omega_R^2 + 2 \omega_I^2 \right)^{-1} \right\}^{-1} \right]. \qquad \ldots(44)$$

Using the condition that $\omega_R m^{-1} < 0$, we conclude that $S_4 (r)$ is positive in the interval $r_1 \leqslant r \leqslant r_2$ if $m^2 V_\theta^2 r^{-2} \geqslant 3 \omega_R^2$.

Further, if $|m| > 1$,

$$S_5 (r) > \frac{m^2 V_\theta^2}{|\omega|^2 r^2} \left[\frac{\beta^2}{4} - \frac{\beta}{2 r} \left(\frac{5 m^2 + 3 n^2 r^2}{m^2 + n^2 r^2} \right) + \frac{n^2}{m^2} (m^2 - 4) \right]$$

and if $|m| = 1$,

$$S_5 (r) > \frac{V_\theta^2}{|\omega|^2 r^2} \left[\frac{\beta^2}{4} - 5 \left(\frac{\beta}{2r} + n^2 \right) \right].$$

We conclude that the conditions necessary for the amplification of modes such that $m^2 V_\theta^2 \geqslant 3 r^2 \omega_R^2$ everywhere, the magnetic field and density distributions must be such that

$$\left(r + \frac{\beta m^2}{2 n^2} \right) \left(\frac{V_\theta^2}{r^2} \right)' - N^2 \left(1 + \frac{m^2}{n^2 r^2} \right)$$

$$> \begin{cases} \dfrac{m^2 V_\theta^2}{n^2 r^2} \left[\dfrac{\beta^2}{4} - \dfrac{\beta}{2 r} \left(\dfrac{5 m^2 + 3 n^2 r^2}{m^2 + n^2 r^2} \right) + \dfrac{n^2}{m^2} (m^2 - 4), \right. & \text{if } |m| > 1 \\ \dfrac{V_\theta^0}{n^2 r^2} \left[\dfrac{\beta^2}{4} - 5 \left(\dfrac{\beta}{2 r} + n^2 \right) \right], & \text{if } |m| = 1. \end{cases}$$

somewhere in the fluid.

6. Slow Amplifying Waves

We concentrate on the slow waves characteristic of rapidly rotating fluid with the assumption that radial, axial and azimuthal wave lengths are all of the same order r_*. As

$$\Omega^2 \gg \frac{V_*^2}{r_*^2} + N_*^2$$

the slow modes will have

$$|\omega|^2 \sim \frac{V_*^2\,(V_*^2\ \ r_*^{-2} + N_*^2\ )}{\Omega^2\ \ r_*^2} \qquad \text{(see Acheson and Hide 1973).}$$

Then the coefficient of $|\psi|^2$ in (41) may be replaced with error $O\left[\frac{(V_*^2\ \ r_*^{-2} + N_*^2)\, r_*^2}{\Omega^2\, V_*^2}\right]$

by

$$-\frac{n^2}{|\omega|^2}\left[r\left(\frac{V_\theta^2}{r^2}\right)' + \frac{\beta m^2}{2n^2}\left(\frac{V_\theta^2}{r^2}\right)' - N^2\left(1+\frac{m^2}{r^2 n^2}\right)\right] - \frac{2\,\Omega^2 n^2}{m\,\omega_R}$$
$$\times\left[\frac{m\,\beta}{2n^2 r} + \left(\frac{2r^2+3m^2n^{-2}}{r^2+m^2n^{-2}}\right)\right] + \frac{m^2V_\theta^2}{r^2|\omega|^2}\left[\frac{\beta^2}{4} - \frac{\beta}{2r}\left(\frac{5m^2+3n^2r^2}{m^2+n^2r^2}\right)\right.$$
$$\left. + \left(\frac{1}{r^2}+\frac{n^2}{m^2}\right)(m^2-4) + \left(\frac{1+3m^2/n^2r^2}{r^2+m^2n^{-2}}\right)\right]. \qquad \ldots(45)$$

We require this to be negative somewhere to avoid violation of (41).

Since $\frac{\omega_R}{m} < 0$, for unstable modes $|m| > 1$, this implies

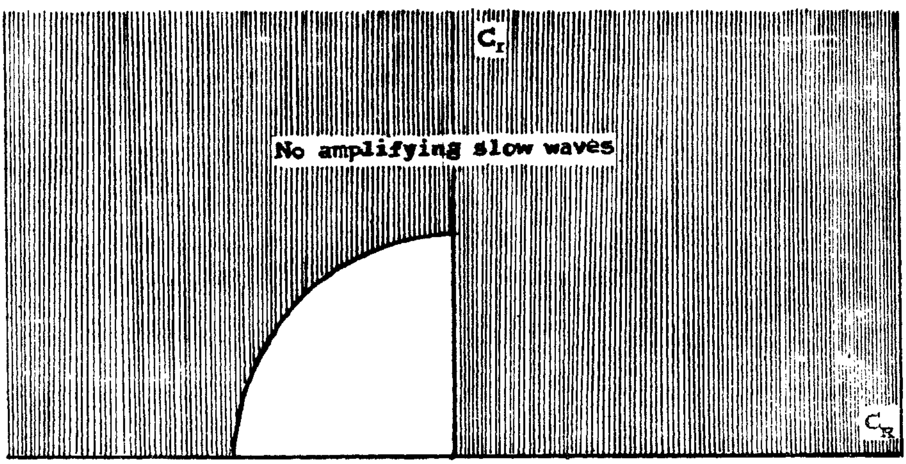

FIG. 1. Illustration shows the quadrant of the complex C plane to which slow amplifying waves are confined when the rotation is uniform and the magnetic field is azimuthal. The radius of the quadrant when $|m| > 1$ is

$$\frac{1}{4|\Omega|}\ \max\left\{ r\left(\frac{V_\theta^2}{r^2}\right)' + \frac{\beta m^2}{2n^2}\left(\frac{V_\theta^2}{r^2}\right)' - N^2\left(1+\frac{m^2}{n^2r^2}\right) - \frac{m^2V_\theta^2}{n^2r^2}\left[\frac{\beta^2}{4} - \frac{\beta}{2r}\left(\frac{5m^2+3n^2r^2}{m^2+n^2r^2}\right) + \frac{n^2}{m^2}\left(m^2-4\right)\right]\right\}.$$

that $4\left|\frac{\Omega}{m\,\omega_R}\right|\left(\omega_R^2+\omega_i^2\right) < \max\left\{r\left(\frac{V_\phi^2}{r^2}\right)' + \frac{\beta\,m^2}{2\,n^2}\left(\frac{V_\phi}{r^2}\right)'\right.$

$$\left. - N^2\left(1+\frac{m^2}{n^2 r^2}\right) - \frac{m^2 V_\phi}{n^2 r^2}\left[\frac{\beta^2}{4} - \frac{\beta}{2r}\left(\frac{5\,m^2+3\,n^2 r^2}{m^2+n^2 r}\right) + \frac{n^2}{m^2}(m^2-4)\right]\right\}$$

which can be written as

$$|C| < \frac{1}{4\,|\Omega|}\max\left\{r\left(\frac{V_\phi^2}{r^2}\right)' + \frac{\beta\,m^2}{2\,n^2}\left(\frac{V_\phi}{r^2}\right)' - N^2\left(1+\frac{m^2}{n^2 r^2}\right)\right.$$

$$\left. - \frac{m^2 V_\phi^2}{n^2 r^2}\left[\frac{\beta^2}{4} - \frac{\beta}{2r}\left(\frac{5\,m^2+3\,n^2 r^2}{m^2+n^2 r^2}\right) + \frac{n^2}{m^2}(m^2-4)\right]\right\} \quad \ldots(46)$$

where $\frac{\beta\,m^2}{2\,n^2}\left(\frac{V_\phi^2}{r^2}\right)' - \frac{m^2 V_\phi^2}{n^2 r^2}\left[\frac{\beta^2}{4} - \frac{\beta}{2r}\left(\frac{5\,m^2+3\,n^2 r^2}{m^2+n^2 r^2}\right)\right] > 0. \quad \ldots(47)$

Thus, for $|m| > 1$ and $C_R\,\Omega < 0$, we find that "the complex wave speed C of any unstable slow mode must lie within the quadrant of the complex C plane".

In comparing the results (46) and (47) with that of Acheson (1973) we conclude that the radius of the quadrant has increased due to the relaxation of Boussinesq approximation which means that density stratification makes the systems more unstable.

7. Conclusion

In the case of axisymmetric disturbances, we generally observe that the presence of magnetic field (which increases with distance from the axis of rotation) has destabilizing influence and vice versa, but the introduction of density stratification ($\beta > 0$) is to reduce the rate of increase of instability. Further, we observe that the density stratification without Boussinesq approximation has more stabilizing influence to axisymmetric disturbances. However the system can be made stable by 'rapid' rotation of the fluid when the magnetic field gradient is moderate.

The analysis of non-axisymmetric disturbances shows that all unstable modes propagate 'westward', regardless of the details of both magnetic field and density profiles. These non-axisymmetric disturbances generate slow hydromagnetic waves in a uniformly rotating fluid and the relaxation of Boussinesq approximation makes the system more unstable.

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