Indian J. pure appl. Math., 13(9) : 1075–1084, September 1982

BASIC PROPERTIES OF A NEW TYPE OF POLYNOMIAL SET

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(*Received* 24 *February* 1981; *after revision* 2 *September* 1981)

In an attempt to provide an elegant generalization of the polynomial set $D_n(x; a, k)$, we have introduced a sequence $\left\{ P_n^{(\alpha,\beta)}(x, y; a, k) \right\}_{n=0}^{\infty}$. The present paper is concerned with series representation, recurrence relations and asymptotic behaviour of $P_n(x, y)$. At the end, the corresponding numbers P_n are also defined and the characterisation property is derived.

1. INTRODUCTION

The sequence of polynomials $\left\{ D_n(x; a, k) \right\}_{n=0}^{\infty}$ was defined by Karande and Thakare (1975), by means of the generating relation

$$\frac{2\,(t/2)^k\, e^{xt}}{e^t - a} = \sum_{n=0}^{\infty} D_n(x; a, k)\, \frac{t^n}{n!}. \qquad \ldots(1.1)$$

Later on many mathematicians have generalized the above sequence of polynomials in different ways. Some examples are as follows:

I. Srivastava (1979, p. 152) has generalized the sequence of polynomials $\left\{ D_n(x; a, k) \right\}_{n=0}^{\infty}$ by means of the generating relation

$$\frac{c^3\,(t^3/c^2)^k \exp\,(xt + yt^2)}{(\exp\,(t)-a)\,(\exp\,(t^2)-b)} = \sum_{n=0}^{\infty} A_n(x, y; a, b, c, k)\, \frac{t^n}{n!}. \qquad \ldots(1.2)$$

Many properties of the polynomial set $A_n(x, y; a, b, c)$ are given by Srivastava (1979).

II. Chaubey (1980, p. 121) has further generalized the result (1.2) as

$$\frac{h^{l+m}\, t^{l+mm'}\,(1 + wt)^{\mu x/w}\,(1 + wt^{m'})^{\nu y/w}}{[(1 + wt)^{h/w} - 1]^l\,[(1 + wt^{m'})^{h/w} - 1]^m} = \sum_{n=0}^{\infty} B_{n;h,w}^{l,m,m'}(x,y)\, \frac{t^n}{n!}. \qquad \ldots(1.3)$$

The aim of the present paper is to generalize the result (1.1) in an entirely new way indicated in (2.1). The connection between (1.1) and (2.1) is given in (2.2). A number of properties like differential and mixed recurrence relations, asymptotic behaviour and some other interesting results are also obtained. The paper is concluded by defining the numbers P_n and deriving the characterisation.

2. Definition

We define the sequence $\left\{P_n^{(\alpha,\beta)}(x, y; a, k)\right\}_{n=0}^{\infty}$ by means of the generating relation

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x, y; a, k)\frac{t^n}{n!} = \frac{2\,(t/2)^k\,(1-xt)^{-\alpha}}{(1-y)t^{-\beta}-a} \qquad \ldots(2.1)$$

where a, α and β are real numbers, k is a non-negative integer, and (for convergence) $|t| < \min\{|x|^{-1}, |y|^{-1}\}$.

The set $P_n^{(\alpha,\beta)}(x, y; a, k)$ defined by (2.1) is a generalization of the sequence of polynomial $\left\{D_n(x; a, k)\right\}_{n=0}^{\infty}$ given by Karande and Thakare (1975) for

$$\lim_{\min\{|\alpha|, |\beta_1\}\to\infty}\left\{P_n^{(\alpha,\beta)}(x/\alpha.\ 1/\beta; a, k)\right\} = D_n(x; a, k),\ n \geqq 0. \qquad \ldots(2.2)$$

Note : In the remaining paper $P_n^{(\alpha,\beta)}(x, y; a, k)$ will be written as $P_n(x, y)$ for simplicity and then $P_n(x, y; a+1)$ will mean $P_n^{(\alpha,\beta)}(x, y; a+1, k)$.

3. Series Representation

$$\sum_{n=0}^{\infty} P_n(x, y)\frac{t^n}{n!} = \frac{2\,(t/2)^k\,(1-xt)^{-\alpha}}{(1-yt)^{-\beta}-a}$$

$$= \sum_{n,m,s=0}^{\infty} \frac{(\alpha)_n\, x^n\,(-1)\,2^{1-k}\,(\beta m)_s\, y^s\, t^{n+k+s}}{n!\, s!\, a^{m+1}}$$

$$= \sum_{n,m=0}^{\infty} \frac{(-1)\,2^{1-k}\,(\alpha)_n\, x^n\, t^{n+k}}{n!\, a^{m+1}}\ {}_2F_1\left[\begin{matrix} -n, \beta m \\ \\ 1-\alpha-n\end{matrix}\ ;\ \frac{y}{x}\right].$$

Hence

$$P_n(x, y) = -\frac{(\alpha)_{n-k}\, x^{n-k}\, n!\, 2^{1-k}}{(n-k)!}$$

$$\times \sum_{m=0}^{\infty} a^{-m-1}\ {}_2F_1\left[\begin{matrix} -n+k, \beta m \\ \\ 1-\alpha-n+k\end{matrix}\ ;\ \frac{y}{x}\right],\quad k \leqslant n. \qquad \ldots(3.1)$$

The series on the R.H.S. of (3.1) is convergent if

$$a > c, \text{ where } c = \lim_{m\to\infty}\frac{f(x, y, m+1)}{f(x, y, m)}$$

and $f(x, y, m) = {}_2F_1 \left[\begin{matrix} -n+k, \beta m \\ 1-\alpha-n+k \end{matrix} ; \frac{y}{x} \right].$

4. Recurrbnce Relations

(i) *Differential Recurrence Relation*

(a) From (2.1), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} P_n(x, y) \frac{t^n}{n!} = \sum_{m,n=0}^{\infty} \frac{\alpha x^m}{n!} P_n(x, y)\, t^{n+m+1}.$$

Thus we have

$$\frac{\partial}{\partial x} P_n(x, y) = \sum_{m=0}^{n-1} \frac{\alpha n!\, x^m P_{n-m-1}(x, y)}{(n-m-1)!} \quad \ldots(4.1)$$

Similarly, we can obtain

$$\frac{\partial^2}{\partial x^2} P_n(x, y) = \sum_{m=0}^{n-2} \frac{(\alpha)_2 (2)_m\, n!\, x^m P_{n-m-2}(x, y)}{(n-m-2)!\, m!} \quad \ldots(4.2)$$

and continuing the above process r times, we have

$$\frac{\partial^r}{\partial x^r} P_n(x, y) = \sum_{m=0}^{n-r} \frac{(\alpha)_r (r)_m\, n!\, x^m P_{n-m-r}(x, y)}{(n-m-r)!\, m!}. \quad \ldots(4.3)$$

(b) Again,

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} P_n(x, y) \frac{t^n}{n!} = -\frac{2\,(t/2)^k (1-xt)^{-\alpha}\, \beta t\, (1-yt)^{-\beta-1}}{[(1-yt)^{-\beta}-a]^2}$$

$$= \sum_{n,m,r=0}^{\infty} \frac{\beta P_n(x, y)\, (\beta m+\beta+1)_r\, y^r\, t^{n+r+1}}{n!\, r!\, a^{m+1}}$$

or

$$\frac{\partial}{\partial y} P_n(x, y) = \sum_{m=0}^{\infty} \sum_{r=0}^{n-1} \frac{n!\, \beta\, (\beta m+\beta+1)_r\, y^r P_{n-r-1}(x, y)}{r!\, (n-r-1)!\, a^{m+1}}. \quad \ldots(4.4)$$

(ii) *Mixed Recurrence Relations*

(a) $$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} P_n(x, y) \frac{t^n}{n!} = \frac{2\,(t/2)^k (1-xt)^{-\alpha-1}}{(1-yt)^{-\beta}-a} \cdot \alpha t$$

or

$$\frac{\partial}{\partial x} P_n(x, y) = \alpha n\, P_{n-1}^{(\alpha+1,\beta)}(x, y). \qquad \ldots(4.5)$$

Also

$$\frac{\partial^2}{\partial x^r} P_n(x, y) = \frac{(\alpha)_2\, n!}{(n-2)!} P_{n-2}^{(\alpha+2,\beta)}(x, y) \qquad \ldots(4.6)$$

and

$$\frac{\partial^r}{\partial x^r} P_n(x, y) = \frac{(\alpha)_r\, n!}{(n-r)!} P_{n-r}^{(\alpha+r,\beta)}(x, y). \qquad \ldots(4.7)$$

In particular if $r = n - k$, then

$$\frac{\partial^{n-k}}{\partial x^{n-k}} P_n\ x.\, y = \frac{(\alpha)_{n-k}\, n!\, 2^{i-k}}{(1-a)} = \text{a constant} \qquad \ldots(4.8)$$

Remark : The above result (4.8) shows that $P_n(x, \lambda)$, where λ is a constant is a polynomial of degree $(n - k)$.

(b) $$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} P_n(x, y) \frac{t^n}{n!} = - \sum_{n=0}^{\infty} P_n(x, y) \frac{t^{n+1}}{n!} \frac{\beta(1-yt)^{-\beta-1}}{(1-yt)^{-\beta}-a}$$

$$= -\beta 2^{k-1} \sum_{n,m=0}^{\infty} P_n^{(\alpha,\beta)}(x, y)\; P_m^{(\beta+1,\beta)}(y, y) \frac{t^{n+m+1}}{n!\, m!}$$

or

$$\frac{\partial}{\partial y} P_n(x, y) = -\beta 2^{k-1} \sum_{m=0}^{n} \binom{n}{m} P_{n-m}^{(\alpha,\beta)}(x, y)\; P_m^{(\beta+1,\beta)}(y, y). \qquad \ldots(4.9)$$

5. Other Interesting Relations

a. (i) $$\sum_{n=0}^{\infty} P_n^{(2\alpha,2\beta)}(x, y; a^2, 2k) \frac{t^n}{n!} = \frac{2(t/2)^{2k}(1-xt)^{-2\alpha}}{(1-yt)^{-2\beta}-a^2}$$

$$= \frac{1}{2} \sum_{n,m=0}^{\infty} P_n(x, y)\, P_m(x, y; -a) \frac{t^{n+m}}{n!\, m!}$$

or

$$P_n^{(2\alpha,2\beta)}(x, y; a^2, 2k) = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} P_{n-m}(x, y)\, P_m(x, y; -a). \qquad \ldots(5.1)$$

(ii) On account of (2.1) we can write

$$\frac{2(t/2)^k (1-xt)^{-\alpha}}{(1-yt)^{-\beta}-a} = \frac{2(t/2)^k (1-xt)^{-(\alpha-\gamma)} (1-xt)^{-\gamma}}{(1-yt)^{-\beta}-a}$$

or

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x, y)\frac{t^n}{n!} = \sum_{n,m=0}^{\infty} \frac{P_n^{(\alpha-\gamma,\beta)}(x, y)(\gamma)_m x^m t^{n+m}}{n!\, m!}$$

$$P_n(x, y) = \sum_{m=0}^{n} \binom{n}{m} P_{n-m}^{(\alpha-\gamma,\beta)}(x, y)(\gamma)_m x^m. \qquad \ldots(5.2)$$

$$\text{(iii)} \quad \sum_{n=0}^{\infty} P_n(1+x, y)\frac{t^n}{n!} = \frac{2(t/2)^k (1-\overline{1+x}\, t)^{-\alpha}}{(1-yt)^{-\beta}-a}$$

$$= \frac{2(t/2)^k (1-xt)^{-\alpha}}{(1-yt)^{-\beta}-a}\left\{1-\frac{t}{1-xt}\right\}^{-\alpha}$$

$$= \sum_{n,m=0}^{\infty} \frac{P_n(x, y)(\alpha)_m (1-xt)^{-m} t^{n+m}}{n!\, m!}$$

$$= \sum_{n,m,s=0}^{\infty} \frac{P_n(x, y), (\alpha)_m (m)_s x^s t^{n+m+s}}{n!\, m!\, s!}$$

$$= \sum_{n,m=0}^{\infty} \frac{(\alpha)_m P_n(x, y)\, t^{n+m}}{n!\, m!}\, {}_2F_1\left[\begin{matrix} -m, -m+1 \\ 1-\alpha-m \end{matrix}; -x\right]$$

$$P_n(1+x, y) = \sum_{m=0}^{n} \binom{n}{m} (\alpha)_m P_{n-m}(x, y)\, {}_2F_1\left[\begin{matrix} -m, -m+1 \\ 1-\alpha-m \end{matrix}; -x\right]. \qquad \ldots(5.3)$$

It can also be proved that

$$\text{(iv)} \quad P_n(1+x, y) = \sum_{m=0}^{n} \sum_{s=0}^{n-m} \binom{n}{m+s}\binom{m+s}{m} (\alpha)_m (m)_s x^s P_{n-m-s}(x, y), \qquad \ldots(5.4)$$

(b) The following results are true for $a \neq 0$

$$\text{(i)} \quad \sum_{n=0}^{\infty} P_n(1-x, y)\frac{t^n}{n!} = \frac{2(t/2)^k [1-(1-x)t]^{-\alpha}}{(1-yt)^{-\beta}-a}$$

$$= \frac{2(t/2)^k (1-xt)^{\alpha}}{1-a(1-yt)^{\beta}} \cdot \frac{[1-(1-x)t]^{-\alpha}(1-xt)^{-\alpha}}{(1-yt)^{-\beta}}$$

$$= -\frac{2(t/2)^k (1-xt)^{\alpha}}{a[(1-yt)^{\beta}-a^{-1}]}(1-t)^{-\alpha}(1-yt)^{\beta}\left\{1-\frac{1-x}{1-t}xt^2\right\}^{-\alpha}$$

(equation continued on p. 1080)

$$= -\frac{1}{a}\sum_{n,m,s=0}^{\infty} P_n^{(-\alpha,-\beta)}(x, y; a^{-1}, k)\,(\alpha)_m\,(1-x)^m\,x^m$$

$$\times \frac{(\alpha+m)_s}{m!\,n!\,s!}\sum_{r=0}^{\infty}\frac{(-\beta)_r\,y^r}{r!}\;t^{n+s+r+2m}$$

$$= -\frac{1}{a}\sum_{n,m,s=0}^{\infty} {}_2F_1\left[\begin{matrix} -s,\ -\beta \\ \\ 1-\alpha-m-s\end{matrix};\ y\right]\frac{(\alpha)_{m+s}\,x^m}{m!}$$

$$\times \frac{(1-x)^m\;P_n^{(-\alpha,-\beta)}(x, y; a^{-1}, k)\;t^{n+2m+s}}{n!\,s!}.$$

Thus we finally obtain

$$P_n(1-x, y) = -\frac{1}{a}\sum_{m=0}^{[n/2]}\sum_{s=0}^{n-2m} {}_2F_1\left[\begin{matrix} -s,\ -\beta \\ \\ 1-\alpha-m-s\end{matrix};\ y\right]\frac{n!\,x^m\,(1-x)^m}{m!}$$

$$\times \frac{(\alpha)_{m+s}\;P_{n-s-2m}^{(-\alpha,-\beta,\beta)}(x, y; a^{-1}, k)}{s!\,(n-s-2m)!}\,. \qquad \ldots(5.5)$$

Similarly, the following results can also be proved:

(ii) $$P_n(1-x, y) = -\frac{1}{a}\sum_{s=0}^{n}\sum_{m=0}^{[(n-s)/2]}\frac{(\alpha)_{m+s}\,x^m\,(1-x)^m\,n!}{m!\,s!\,(n-2m-s)!}$$

$$\times\ {}_2F_1\left[\begin{matrix} -s,\ -\beta \\ \\ 1-\alpha-m-s\end{matrix};\ y\right]P_{n-2m-s}^{(-\alpha,-\beta)}(x, y; a^{-1}, k). \qquad \ldots(5.6)$$

(iii) $$P_n(1-x, y) = \frac{1}{a}\sum_{s=0}^{n}\sum_{m=0}^{[s/2]}\binom{n}{s}\frac{(-1)^{m+1}\,(1-x)^m\,x^m\,(-s)_{2m}\,(\alpha)_s}{(1-\alpha-s)_m\,m!}$$

$$\times\ {}_2F_1\left[\begin{matrix} -s+2m,\ -\beta \\ \\ 1-\alpha+m-s\end{matrix};\ y\right]P_{n-s}^{(-\alpha,-\beta)}(x, y; a^{-1}, k). \qquad \ldots(5.7)$$

(iv) $$P_n(1-x, y) = -\frac{1}{a}\sum_{s=0}^{n}\sum_{r=0}^{n-s} {}_2F_1\left[\begin{matrix} -\frac{1}{2}s,\ -\frac{1}{2}s+\frac{1}{2} \\ \\ 1-\alpha-s\end{matrix};\ -4x(1-x)\right]$$

$$\times\ \frac{(\alpha)_s\,(-\beta)_r\,y^r\,n!}{(n-s-r)!\,s!\,r!}\,P_{n-s-r}^{(-\alpha,-\beta)}(x, y; a^{-1}, k) \qquad \ldots(5.8)$$

(v) $$P_n(1-x, y) = -\frac{1}{a} \sum_{s=0}^{n} \sum_{m=0}^{[(n-s)/2]} \sum_{r=0}^{n-s-2m} \frac{n!\,(\alpha)_{s+m}\,(1-x)^m\,x^m\,(-\beta)_r\,y^r}{r!\,m!\,s!\,(n-2m-s-r)!}$$

$$\times\ P^{(-\alpha,-\beta)}_{n-s-r-2m}(x, y; a^{-1}, k). \qquad \ldots(5.9)$$

(vi) $$P_n(1-x, y) = -\frac{1}{a} \sum_{m=0}^{[n/2]} \sum_{s=0}^{n-2m} \sum_{r=0}^{n-2m-s} \frac{(\alpha)_{s+m}\,(1-x)^m\,x^m\,(-\beta)_r\,y^r\,n!}{r!\,s!\,m!\,(n-2m-s-r)!}$$

$$\times\ P^{(-\alpha,-\beta)}_{n-2m-s-r}(x, y; a^{-1}, k). \qquad \ldots(5.10)$$

(vii) $$P_n(1-x, y) = -\frac{1}{a} \sum_{m=0}^{[n/2]} \sum_{r=0}^{n-2m} \frac{(\alpha)_m\,x^m\,(1-x)^m\,n!\,y^r\,(-\beta)_r}{r!\,m!\,(n-r-2m)!}$$

$$\times\ {}_2F_1\left[\begin{matrix}-r,\ \alpha+m \\ 1+\beta-r\end{matrix};\ \frac{1}{y}\right] P^{(\alpha-,\beta)}_{n-r-2m}(x, y; a^{-1}, k). \qquad \ldots(5.11)$$

(viii) $$P_n(1-x, y) = -\frac{1}{a} \sum_{r=0}^{n} \sum_{m=0}^{[r/2]} \frac{(\alpha)_m\,x^m\,(1-x)^m\,n!\,y^{r-2m}\,(-\beta)_{r-2m}}{m!\,(r-2m)!\,(n-r)!}$$

$$\times\ {}_2F_1\left[\begin{matrix}-r+2m,\ \alpha+m \\ 1+\beta-2m-r\end{matrix};\ \frac{1}{y}\right] P^{(-\alpha,-\beta)}_{n-r}(x, y; a^{-1}, k).$$

$$\ldots(5.12)$$

(ix) $$P_n(1-x, y) = -\frac{1}{a} \sum_{r=0}^{n} \sum_{m=0}^{\left[\frac{n-r}{2}\right]} \frac{(\alpha)_m\,x^m\,(1-x)^m\,n!\,y^r\,(-\beta)_r}{r!\,m!\,(n-2m-r)!}$$

$$\times\ {}_2F_1\left[\begin{matrix}-r,\ \alpha+m \\ 1+\beta-r\end{matrix};\ \frac{1}{y}\right] P^{(-\alpha,-\beta)}_{n-2m-r}(x, y; a^{-1}, k). \qquad \ldots(5.13)$$

(x) $$P_n(1-x, y) = -\frac{1}{a} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(\alpha)_s\,(-\beta)_r\,n!\,y^r}{s!\,r!\,(n-s-r)!}\, P^{(-\alpha,-\beta)}_{n-s-r}(x, y; a^{-1}, k)$$

$$\times\ {}_3F_2\left[\begin{matrix}\alpha+s,\ -\frac{1}{2}r,\ -\frac{1}{2}r+\frac{1}{2} \\ \frac{1+\beta-r}{2},\ \frac{2+\beta-r}{2}\end{matrix};\ \frac{x(1-x)}{y^2}\right]. \qquad \ldots(5.14)$$

$$\text{(xi)}\quad P_n(1-x, y) = -\frac{1}{a}\sum_{s=0}^{n}\sum_{r=0}^{n-s}\sum_{m=0}^{\left[\frac{n-s-r}{2}\right]}\frac{(\alpha)_{m+s}\, x^m (1-x)^m\, n!\, (-\beta)_r\, y^r}{m!\, n!\, s!\, (n-s-r-2m)!}$$

$$\times\ P^{(-\alpha,-\beta)}_{n-s-r-2m}(x, y; a^{-1}, k). \qquad \text{..(5.15)}$$

(c) On account of (3.1) we have

$$\text{(i)}\quad P_n(x,y) = \frac{(\alpha)_{n-k}\, x^{n-k}\, 2^{1-k}\, n!}{(-a)(n-k)!}\sum_{m=0}^{\infty}\frac{1}{a^m}\, {}_2F_1\left[\begin{matrix}-n+k,\ \beta m\\ \ \\ 1-\alpha-n+k\end{matrix};\ y/x\right]$$

$$= (xy)^{n-k}\,\frac{(\alpha)_{n-k}\, y^{k-n}\, 2^{1-k}\, n!}{(-a)\ (n-k)!}\sum_{m=0}^{\infty}\frac{1}{a^m}\, {}_2F_1\left[\begin{matrix}-n+k,\ \beta m\\ \ \\ 1-\alpha-n+k\end{matrix};\ \frac{x^{-1}}{y^{-1}}\right].$$

or

$$P_n(x,y) = (xy)^{n-k}\, P_n(y^{-1}, x^{-1}). \qquad \text{...(5.16)}$$

(ii) The following result can also be deduced

$$P_n(x, y^{-1}) = (x/y)^{n-k}\, P_n(y, x^{-1}) \qquad \text{...(5.17)}$$

6. Asymptotic Behaviour

Theorem—Prove that

$$\lim_{n\to\infty}\left\{\frac{x^{-n}\, P_n(x,y)}{(\alpha)_n}\right\} = \frac{2^{1-k}\, x^{-k}}{(1-yx^{-1})^{-\beta}-a} \qquad \text{...(6.1)}$$

where

$$\text{(i)}\quad \left|\frac{(-n)_{p+k}}{(1-\alpha-n)_{p+k}}\right| \leqslant 1 \qquad \text{...(6.2)}$$

and

$$\text{(ii)}\quad \frac{2^{1-k}\, x^{-k}}{(1-yx^{-1})^{-\beta}-a} \qquad \text{...(6.3)}$$

is convergent.

Proof : Due to the given condition (i)

$$\left|\frac{(-n)_{p+k}\, 2^{1-k}\, (\beta m)_p\, y^p\, x^{-p-k}}{(1-\alpha-n)_{p+k}\, p!\, a^{m+1}}\right| \leqslant \left|\frac{2^{1-k}\, (\beta m)_p\, y^p\, x^{-p-k}}{p!\, a^{m+1}}\right|$$

and due to condition (ii)

$$\sum_{m,p=0}^{\infty}\frac{(-1)\, 2^{1-k}\, (\beta m)_p\, x^{-p-k}\, y^p}{a^{m+1}\, p!}$$

isconvergent.

Hence with the help of Tannery's theorem (Bromwich 1926, p. 136) we can apply the limit $n \to \infty$ on the L.H.S. of (6.1). This

$$\lim_{n\to\infty} \left\{ \frac{x^{-1} P_n (x, y)}{(\alpha)_n} \right\} = \lim_{n\to\infty} \left\{ \sum_{m=0}^{\infty} \sum_{p=0}^{n-k} \frac{x^{-n}\, n!\, 2^{1-k} (\beta m)_p}{(\alpha)_n\, a^{m+1}} \times \frac{(-1)\,(\alpha)_{n-k}\, x^{n-k-p} (-n+k)_p\, y^p}{(n-k)!\, p!\, (1-\alpha-n+k)_p} \right\}$$

$$= \sum_{m,p=0}^{\infty} \frac{(-1)\, 2^{1-k} (\beta m)_p\, y^p\, x^{-p-k}}{p!\, a^{m+1}}$$

i.e.

$$\lim_{n\to\infty} \left\{ \frac{x^{-n} P_n (x, y)}{(\alpha)_n} \right\} = \frac{2^{1-k} x^{-k}}{(1-y\, x^{-1})^{-\beta} - a}$$

Hence the theorem.

7. Definition

We define the P_n numbers for $x = 0$ and $y = 1/\beta$ by means of the generating relation

$$\sum_{n=0}^{\infty} P_n \frac{t^n}{n!} = \frac{2\,(t/2)^k}{(1-t/\beta)^{-\beta} - a}\,. \qquad \ldots(7.1)$$

Application

$$\lim_{|\beta|\to\infty} \sum_{n=0}^{\infty} P_n \frac{t^n}{n!} = \lim_{|\beta|\to\infty} \frac{2\,(t/2)^k}{(1-t/\beta)^{-\beta} - a}$$

$$= \frac{2\,(t/2)^k}{e^t - a}\,, \qquad \ldots(7.2)$$

Now, when $k = a = 1$, P_n becomes the Bernoulli numbers B_n (Rainville; 1960, p. 300, (4)).

8. Characterisation

On account of (2.1), we have

$$\sum_{n=0}^{\infty} P_n (x+z, y) \frac{t^n}{n!} = \frac{2\,(t/2)^k\, (1-\overline{x+z}\; t)}{(1-yt)^{-\beta} - a}$$

$$= \sum_{n,r,m=0}^{\infty} \frac{(\alpha)_r\, x^r\, (r)_m\, z^m\, P_n (z, y)\, t^{n+m+r}}{n!\, r!\, m!}\,.$$

Comparing the coefficients of t^n, we get

$$P_n(x+z, y) = \sum_{r=0}^{n} \binom{n}{r} (\alpha)_r\, x^r\, {}_2F_1\left[\begin{matrix} -r, & -r+1 \\ & 1-\alpha-r \end{matrix};\ \frac{-z}{x}\right] P_{n-r}(z, y). \qquad \ldots(8.1)$$

Also

$$P_n(x+z, y) = \sum_{r=0}^{n} \sum_{m=0}^{n-r} \frac{(\alpha)_r\, (r)_m\, n!\, x^r\, x^r\, z^m\, P_{n-m-r}(z, y)}{r!\, m!\, (n-m-r)!}. \qquad \ldots(8.2)$$

Putting $z = 0$ and $y = 1/\beta$ in any of the above expression we get

$$P_n(x, 1) \doteq (x+P)^n \qquad \ldots(8.3)$$

An Alternative Method

$$\sum_{n=0}^{\infty} P_n(x, 1) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(\alpha)_{n-r}\, x^{n-r}\, P_r\, t^n}{(n-r)!\, r!}$$

or

$$P_n(x, 1) = \sum_{r=0}^{n} \binom{n}{r} (\alpha)_{n-r}\, x^{n-r}\, P_r; \qquad \ldots(8.4)$$

thus $P_n(x, 1) \doteq (x+P)^n$.

Acknowledgement

The authors are grateful to Prof. H. M. Srivastava (University of Victoria, Canada) for his valuable suggestions for the improvement of the paper.

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