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# VIBRATIONAL RELAXATION EFFECTS ON THE CURVATURE OF AN ATTACHED SHOCK WAVE

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A pointed obstacle is assumed placed with respect to a uniform supersonic flow ahead of it. It is assumed that an oblique shock wave attached to the leading edge of the obstacle appears from the vertex so that the flow after the shock is along the surface of the obstacle. The effects of vibrational relaxation on downstream propagation of a shock wave attached to the leading edge of an obstacle in a steady plane flow are studied here. Also an interesting relation between the curvature of the attached shock wave and that of a stream line is established.

## 1. INTRODUCTION

In dealing with problems of hypersonic flights at high altitude, temperatures of many thousands degrees of Kelvin can easily be attained during flight, as the kinetic energy of the re-entering craft is dissipated by the atmospheric gas through shock compression and viscous heating. The air molecules, atoms and other species, after absorbing this kinetic energy, may go through a change of chemical composition. In the temperature range from 1000°K to 3000°K only chemical reaction involved is that of vibrational excitation.

The non-equilibrium flows past a blunt body have been studied by Freeman (1958), Lick (1960) and many others. Sedney and Gerber (1967) determined the shock curvature and gradients of flow variables behind a shock attached to pointed axisymmetric bodies in non-equilibrium flows. Recently, Ram and Sharma (1973) treated the problem of regular reflection of an oblique shock in a plane flow of an ideal dissociating gas. In this paper we have studied the downstream effects of the propagation of a shock wave attached to the leading edge of an obstacle in a steady plane flow with vibrational relaxation. We have also established an interesting relation between the curvature of the attached shock wave and that of a stream line at the vertex of a pointed obstacle. Some interesting results have also been deduced and discussed.

## 2. BASIC EQUATIONS

The basic equations governing the two-dimensional steady flow for a simple model of a vibrating gas are,

$$\rho u_{i,i} + u_i \rho_{,i} = 0, \qquad \ldots(2.1)$$

$$\rho u_j u_{i,j} + p_{,i} = 0, \qquad \ldots(2.2)$$

$$u_i \sigma_{,i} + \rho \phi (T) (\sigma - \bar{\sigma}) = 0, \qquad \ldots(2.3)$$

$$\rho u_i h_{,i} - u_i p_{,i} = \frac{\rho\phi (T) (\sigma - \bar{\sigma})}{T} \qquad \ldots(2.4)$$

where $p$, $u_i$, $\rho$ and $h$ are respectively the pressure, the velocity components, the density and the enthalpy of the system. $\phi (T)$ is the vibrational frequency and $\bar{\sigma}$ is the local equilibrium value of the vibrational energy $\sigma$.

In consequence of (2.1) and (2.3), eqn. (2.4) can be written in the form

$$u_i p_{,i} + c^2 \rho u_{i,i} + \frac{\rho^2}{T} (\gamma - 1) \phi (T) (\bar{\sigma} - \sigma) = 0, \qquad \ldots(2.5)$$

where $c^2 = \gamma p/\rho$ is the square of the local speed of sound and $\gamma$ is the heat exponent of the vibrating gas.

## 3. Curvature of an Attached Shock Wave

Let us consider that a pointed obstacle with vertex at $V$ is placed symmetrically with respect to a uniform supersonic flow in front of it. Let $\theta$ be the angle which the tangent to the obstacle at $V$ makes with the direction of the flow which is parallel to $x_1$-axis. Let $\phi$ be the angle which the tangent to the shock makes with the direction of upstream flow. When the flow reaches the obstacle at $V$, it will deflect through an angle $\theta$. Now if the angle $\theta$ is less than the corresponding maximum angle of deflection across an oblique shock, the flow conditions may be represented by an oblique shock attached to the leading edge of the obstacle so that the flow after the shock is along the surface of the obstacle (Pai 1959).

Let the shock configuration in a two-dimensional steady flow be given by

$$x_i = x_i (s), \qquad \ldots(3.1)$$

where $x_i$ $(i = 1,2)$ are the rectangular cartesian coordinates of a point $P$ on the shock and the parameter $s$ measures the arcual distance along the shock curve. If $\lambda_i$ and $n_i$ respectively are the components of unit tangent and unit normal vectors to the shock at the point $V$, then we have

$$\partial_s x_i = \lambda_i, \ \partial_s \lambda_i = K n_i, \ \partial_s n_i = -K\lambda_i, \qquad \ldots(3.2)$$

where $\partial_s$ is the operator of differentiation w. r. t. $s$ and $K$ is the curvature of the attached shock wave at the point $V$. We consider the propagation of a shock wave normal to itself. Let the jump in any quantity $Z$ across the shock be denoted by $[Z] = Z - Z_1$ where $Z$ and $Z_1$ are values of $Z$ just behind and just in front of the shock respectively. The geometrical compatibility condition of first order for the study of discontinuities in the continuum mechanics (Thomas 1957) is

$$[Z_{,i}] = [\partial_n Z] n_i + \partial_s [Z] \lambda_i, \qquad \ldots(3.3)$$

where

$$\partial_n z \ \underline{\underline{\text{def}}} \ n_i Z_{,i}.$$

If we define the density strength $\delta$ of the shock by the relation $\delta = [\rho]/\rho_1$, we have the follwing jump relations (Ram 1968)

$$\left.\begin{aligned} [u_i] &= -\delta(1+\delta)^{-1}\, u_{1n}\, n_i \\ [p] &= \delta \rho_1 u^2_{1n}\,(1+\delta)^{-1} \end{aligned}\right\} \qquad \ldots(3.4)$$

where

$$\delta = \frac{2(\rho_1 u^2_{1n} - \gamma p_1)}{2\gamma p_1 + (\gamma - 1)\rho_1 u^2_{1n}}. \qquad \ldots(3.5)$$

The position of the attached shock at the vertex $V$ is shown in FIG. 1.

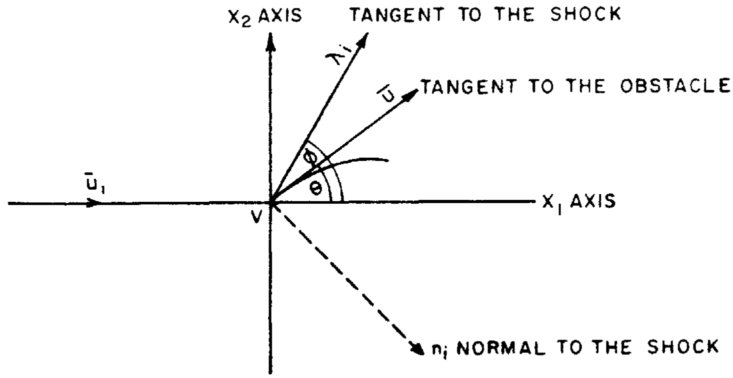


FIG. 1. Position of an attached shock wave at the vertex $V$ of the pointed obstacle.

It can be easily shown that the strength of the shock depends upon the wave angle $\phi$, i. e. the angle which the free stream makes with the tangent to the shock. From (3.4) and the Fig. 1, we have

$$\frac{u_n}{u_{1n}} = \cot\phi \tan(\phi - \theta) = (1+\delta)^{-1} \qquad \ldots(3.6)$$

hence, we get

$$\delta = \sin\theta \sec\theta \operatorname{cosec}(\phi - \theta). \qquad \ldots(3.7)$$

The relation (3.7) shows that if a pointed obstacle with given wedge angle $\theta$ is placed symmetrically with respect to the upstream flow, the strength $\delta$ of the attached shock wave at $V$ depends only on the wave angle $\phi$. Also from (3.7) we can conclude the following interesting results :

(i) The physical appearance of an attached shock wave is possible only when $\phi$ satisfies the inequality $\theta < \phi < \pi/2$.

(ii) In order to reduce the strength of the attached shock to the minimum, the semi-vertex angle $\theta$ of the pointed obstacle should satisfy the condition $\phi = \frac{1}{2}(\theta + \frac{1}{2}\pi)$.

The importance of these results can be very much realised in practical aeronautical problems. In fact, the strong shock waves may cause sudden change in the aerodynamic behaviour of high speed Jet-aircrafts which affects their balance, stability and control by producing undesirable vibrations. The appearance of an attached shock is frequently observed by Jet-aircraft pilots Cambel and Jennings (1958). The wave angle $\phi$ can be measured by optical methods. Thus by employing observation techniques and modelling the Jet-aircraft accordingly, the shock compression effects in the gas due to an attached shock wave can be minimized. The variation of $\delta$ along the shock curve is given by

$$\partial_s \delta = -\xi K, \qquad \ldots(3.8)$$

where

$$\xi = \{\sin\theta \cos(2\phi - \theta)\}\ \{\cos\phi \sin(\theta - \phi)\}^{-2}.$$

Differentiating (3.4) along the shock curve under the assumption of uniform flow in front of the shock and using (3.2) and (3.8), we get

$$\left.\begin{aligned} \partial_s p &= -2(1+\delta)^{-1}\,\delta K \rho_1 u_{1n} u_{1t} - (1+\delta)^{-2}\,\xi K \rho_1 u^2_{1n} \\ \partial_s u_i &= K\delta(1+\delta)^{-1}\,(u_{1t} n_i + u_{1n}\lambda_i) + \xi K u_{1n}(1+\delta)^{-2} n_i \end{aligned}\right\}. \qquad \ldots(3.9)$$

By virtue of (3.3), the eqns. (2.1), (2.2) and (2.5) can be transformed into the following forms:

$$\rho n_i \partial_n u_i + \rho\lambda_i \partial_s u_i + u_n \partial_n \rho + u_t \partial_s \rho = 0, \qquad \ldots(3.10)$$

$$\rho u_n \partial_n u_i + \rho u_t \partial_s u_i + (\partial_n p) n_i + \lambda_i \partial_s p = 0, \qquad \ldots(3.11)$$

$$u_n \partial_n p + u_t \partial_s p + \rho c^2 \{n_i \partial_n u_i + \lambda_i \partial_s u_i\} + (\gamma - 1) F = 0, \qquad \ldots(3.12)$$

where $F = \rho^2 \phi(T)\,(\bar{\sigma} - \sigma)/T$.

Multiplying (3.11) by $n_i$ and using (3.12), we get

$$n_i \partial_n u_i = \{u_t \partial_s p + \rho c^2 \lambda_i \partial_s u_i - \rho n_i u_t u_n \partial_s u_i + (\gamma - 1)F\}\{\rho u^2_n - \rho c^2\}^{-1} \qquad \ldots(3.13)$$

Multiplying (3.11) by $\lambda_i$, we get

$$\rho u_n \lambda_i \partial_n u_i = -\{\rho\lambda_i u_t \partial_s u_i + \partial_s p\}. \qquad \ldots(3.14)$$

If $Q$ stands for the curvature of a stream line, we have (Ram 1968)

$$Q = -V^{-3} e_{ik} u_k u_j u_{i,j},$$

where

$$V^2 = u_i u_i,\ e_{11} = e_{22} = 0,\ e_{12} = -e_{21} = 1.$$

The curvature $Q$ at the rear of the shock at $V$ is thus given by

$$Q = V^{-3}(u_n u_t n_i \partial_n u_i - u^2_n \lambda_i \partial_n u_i + u^2_t n_i \partial_s u_i - u_t u_n \lambda_i \partial_s u_i). \qquad \ldots(3.15)$$

Substituting from (3.9), (3.13) and (3.14) in (3.15), we obtain

$$\begin{aligned} Q - \eta F \cos\phi = K\eta u_1 \cos^2\phi [ &-\tfrac{3}{2}\,\delta(1+\delta)^{-1} \rho_1 u^2_1 \sin 2\phi - (1+\delta)^{-2}\,\xi\rho_1 u^2_1 \sin^2\phi \\ &-2\delta(1+\delta)^{-1} \rho_1 M c^2 \tan\phi - (1+\delta)^{-2}\,\xi\rho_1 M c^2 \tan^2\phi \\ &+\delta\rho M c^2 \cot\phi + \rho\delta(1+\delta)^{-1} c^2 \tan\phi + \xi(1+\delta)^{-1}\rho M c^2 \\ &-\xi\rho(1+\delta)^{-3}\, u^2_1 \sin^2\phi], \end{aligned} \qquad \ldots(3.16)$$

where

$$\eta = \frac{u^2_1 \sin\phi}{\rho M c^2 (1+\delta) V^3} \quad \text{and} \quad M = \left(\frac{u^2_n}{c^2} - 1\right).$$

Equation (3.16) provides us an interesting relation between the curvature of an attached shock wave and that of a stream line at the vertex $V$ of the pointed obstacle.

The relation (3.16) shows that in a steady flow of a gas in vibrational excitation, an attached shock at the vertex of a two-dimensional wedge becomes curved due to vibrational relaxation. The shock curvature $K$ at the vertex $V$ is given by

$$K = F\,(u_1 \cos\phi)^{-1}\,[\,\frac{3}{2}\;\delta\,(1+\delta)^{-1}\,\rho_1 u^2{}_1 \sin 2\phi + (1+\delta)^{-2}\,\xi\rho_1 u_1{}^2 \sin^2\phi$$
$$+ 2\delta\,(1+\delta)^{-1}\,\rho_1 M c^2 \tan\phi + (1+\delta)^{-2}\,\xi\rho_1 M c^2 \tan^2\phi$$
$$- \delta\rho M c^2 \cot\phi - \rho\delta\,(1+\delta)^{-1}\,c^2 \tan\phi - \xi(1+\delta)^{-1}\,\rho c^2 M$$
$$+ \xi\rho(1+\delta)^{-3}\,u^2{}_1 \sin^2\phi]^{-1}.$$

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