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FREE CONVECTION FLOW OF AN ELASTICO-VISCOUS FLUID BETWEEN TWO PARALLEL WALLS IN THE PRESENCE OF HEAT SOURCE

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Free convection flow and heat transfer of Truesdell visco-elastic liquid with a heat source in channels with constant wall temperature have been studied. Perturbation method has been employed to find the velocity and temperature distributions in the boundary layer. Effects of elasticity (S) and ratio of the temperature of the planes (m) on the velocity and temperature distributions have been studied. Results are presented graphically. Effect of elasticity is to increase the third order perturbation velocity with increase in m. First order and second order perturbation velocity fields are not affected by the elasticity present in the fluid and decrease with increase in m. It is also evident from that first and second order perturbation temperatures increase with increase in m and third order perturbation decreases with increase in m and is not affected by the elasticity of the fluid. Velocity and temperature fields both increase with increase in buoyancy force parameter (M) and heat source parameter (ξ).

NOMENCLATURE

x, y = distance measured along and perpendicular to the parallel walls

ϕ = $T_j^i\, d_i^j$ = dissipation function

Q = heat added by heat source

c, k = Specific heat capacity and thermal conductivity

ρ, T, p = Density, temperature and pressure of the medium

T_j^i = stress tensor

d_j^i = $1/2\,(V_{i,j} + V_{j,i})$ = strain tensor

S = Visco-elastic parameter

M = buoyancy force parameter

ξ = heat source parameter

μ, μ_c = coefficient of viscosity and coefficient of cross viscosity

λ = relaxation time parameter

Pr, Gr = Prandtl number, Grashoff number

β = Coefficient of thermal expansion

m = ratio of the temperatures of the planes

U_i = $(i = 1, 2, 3)$ = first, second and third order perturbation velocity fields

T_i = $(i = 1, 2, 3)$ = first, second and third order perturbation temperature fields

N_0, N_1 = Nusselt numbers at the planes

s, w = Script s denotes hydrostatic state and script w denotes the conditions at the planes

f_x, f_y = generating body force per unit mass

h = distance between the parallel planes

q = heat transfer from the planes

1. Introduction

In case of free-convection flow, density is considered variable only in forming the buoyancy force. Such a natural flow exists around a vertical hot plate or around a horizontal cylinder. Ostrach (1952, 1954, 1955, 1957) has studied the free-convection flow of a viscous liquid in a vertical channel formed by two parallel planes kept either at a constant temperature or at temperature linearly varying along the planes. The increasing emergence of non-Newtonian fluids, such as moltan plastics, pulps, emulsions etc. as important raw material and products in a large variety of industrial processes, has stimulated a considerable amount of interest in the behaviour of such fluids in motion. Considerable attention has been given in recent years to the study of non-Newtonian fluids and their related transport processes. Some investigations concerning natural connective heat transfer have been reported in literature for non-Newtonian fluids which are of great importance in chemical process industries.

Jain (1962) has studied problem similar to Ostrach (1952-1957) for a visco-elastic fluid proposed by Oldroyd. He pointed out that the effect of elastic number is to increase the central temperature. The Nusselt number is also appreciably altered by the presence of visco-elastic elements. Mishra and Dash (1974) have studied the laminar free-convection flow and heat transfer of a liquid whose viscosity is a function of the flow invariants.

The present paper has been devoted to the study of heat transfer in free-convection flow of Truesdell visco-elastic fluid between two parallel walls in the presence of heat source. Truesdell visco-elastic fluid differs from Oldroyd fluid (Jain 1962) in constitutive equations. The constitutive equation of the former fluid involves stress relaxation time parameter and cross-viscosity whereas the constitutive equation of the latter fluid involves stress relaxation time and stress retardation time alongwith other scalar physical quantities having dimensions of time. Figures (1)–(8) for velocity and temperature distributions have been plotted for different values of heat source parameter, ratio of temperature of the walls and the viscoelastic parameter. This paper can be looked upon as an extension of the work of Ostrach to viscoelastic liquid by taking into account a heat source.

Due to its wide application in chemical engineering, electronics, atomic power and aeronautics the process of natural convection flow has drawn the attention of several authors. The constitutive equation for Truesdell (1955) viscoelastic fluid (1955) is given by

$$T^i_j + \lambda \overline{T}^i_j = 2\mu d^i_j + 4\mu_c d^i_k \; d^k_j , \qquad \ldots(1)$$

where

$$\overline{T}^i_j = \frac{\partial}{\partial t} \; T^i_j + T^i_{j,k} V^k - T^k_i V_{j,k} + T^i_j V^k_{,k} - T^k_j V^i_{,k} .$$

2. Formulation of the Problem

Equations of motion governing the flow for the present configuration are:

$$\frac{\partial p}{\partial x_i} + \rho \left(\frac{\partial V^i}{\partial t} + V^i_{,j} V^j \right) = T^i_{j,j} + \rho f_i, \qquad \ldots(2)$$

$$\rho_c \left(\frac{\partial T}{\partial t} + V_j T_{,j} \right) = k \, T_{,jj} + Q + \phi , \qquad \ldots(3)$$

$$V_{1,1} = 0, \qquad \ldots(4)$$

where , denotes covariant differentiation.

We take x-axis along one plane and y-axis normal to it. The other plane parallel to $y = 0$ is situated at $y = h$.

The stresses and boundary layer equations are given by

$$T_{xx} = 2\lambda \, T_{xy} \frac{\partial u}{\partial y} + \mu_c \left(\frac{\partial u}{\partial y} \right)^2 \qquad \ldots(5)$$

$$T_{xy} = \lambda \, T_{yy} \frac{\partial u}{\partial y} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \qquad \ldots(6)$$

$$T_{yy} = \mu_c \left(\frac{\partial u}{\partial y} \right)^2 \qquad \ldots(7)$$

$$0 = - \frac{\partial p}{\partial x} + \rho f_x + 3\lambda\mu_c \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial y^2} \qquad \ldots(8)$$

$$0 = - \frac{\partial p}{\partial y} + \rho f_y + 2\mu_c \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \qquad \ldots(9)$$

$$3\lambda\mu_c \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p_D}{\partial x} - \beta\rho f_x \theta = 0 \qquad \ldots(10)$$

$$k \frac{\partial^2 T}{\partial y^2} + Q + \left[\lambda\mu_c \left(\frac{\partial u}{\partial y} \right)^4 + \mu \left(\frac{\partial u}{\partial y} \right)^2 \right] = 0 \qquad \ldots(11)$$

$$k \frac{\partial^2 \theta}{\partial y^2} + Q + \left(\frac{\partial u}{\partial y} \right)^2 \left[\lambda\mu_c \left(\frac{\partial u}{\partial y} \right)^2 + \mu \right] = 0. \qquad \ldots(12)$$

The physical significance of λ is that if the motion is stopped suddenly, the stresses will decay as $\exp(-t/\Lambda)$.

Now following Ostrach (1952), the body force term in (8) can be expressed as buoyancy term. The subscript s denotes the hydrostatic condition. Then eqn. (8) gives

$$\rho_s f_x - \frac{\partial p_s}{\partial x} = 0$$

and hence

$$\rho f_x - \frac{\partial p}{\partial x} = - \rho\beta f_x\theta - \frac{\partial p_D}{\partial x} \qquad \text{...(13)}$$

where

$$\theta = T - T_s,\ p_D = p - p_s,\ \beta = - \frac{(\rho - \rho_s)}{\rho (T - T_s)}$$

and $\frac{dp_D}{dy} = \mu_c \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2$ (which follows from equation 9).

Here T_s is the temperature in the hydrostatic state, f_x is the constant which coincides with the acceleration due to gravity. As u and θ are the functions of y alone, it is evident that dp_D/dx is constant. Hence, the pressure gradient dp/dx inside the channel differs from the hydrostatic pressure gradient by atmost a constant. Following Ostrach (1952), the equation (10) can be written as

$$3\lambda\mu_c \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial y^2} = \beta\rho f_x \theta. \qquad \text{...(14)}$$

The boundary conditions of the problem accordingly reduce to

$$\left.\begin{aligned} u(0) &= 0 = u(h) \\ \theta(0) &= T_W(0) - T_S = \theta_W(0) \\ \theta(h) &= T_W(h) - T_S = \theta_W(h) \end{aligned}\right\}. \qquad \text{...(15)}$$

Adopting the following transformations

$$\left.\begin{aligned} &\eta = y/h,\ m = \theta_w(h)/\theta_w(0),\ S = \lambda\mu_c/k^2\mu h^6\ \beta\rho f_x. \\ &\xi = \frac{Qh^2}{k\theta_w(0)},\ u = kU/\rho\beta f_x h^2,\ \theta = \theta_w(0)/M,\ Pr = \frac{c\mu}{k} \\ &Gr = \frac{\rho\beta f_x\, h^4\, \theta_w(0)}{\mu^2},\ M = Pr\ Gr\ \beta\rho f_x \mid c, \end{aligned}\right\} \qquad \text{...(16)}$$

eqns. (14) and (12), using (16), now become

$$U'' + 3SU''(U')^2 - T = 0 \qquad \text{...(17)}$$

$$T'' + (U')^2\, S\,(U'')^4 + \xi M = 0 \qquad \text{...(18)}$$

where the dashes denote differentiation with respect to η.

In view of (16) the boundary conditions (15) reduce to

$$\left.\begin{aligned} U(0) &= 0 = U(1) \\ T(0) &= M,\ T(1) = m\,M. \end{aligned}\right\} \qquad \text{...(19)}$$

Following Agarwal and Upmanyu (1976) expressions for $U(\eta)$ and $T(\eta)$ are assumed in the form :

$$\left.\begin{aligned} U(\eta) &= MU_1 + M^2U_2 + M^3U_3 + \ldots. \\ T(\eta) &= MT_1 + M^2T_2 + M^3T_3 + \ldots. \end{aligned}\right\} \qquad \text{...(20)}$$

The boundary conditions (19) now become :

$$\left.\begin{array}{l} \text{At } \eta = 0,\ U_1 = 0 = U_2 = U_3 \\ \qquad T_1 = 1,\ T_2 = 0 = T_3 \\ \text{At } \eta = 1,\ U_1 = 0 = U_2 = U_3 \\ \qquad T_1 = m,\ T_2 = 0 = T_3 \end{array}\right\} . \quad \ldots(21)$$

Solving (17) and (18) after substituting for $U(\eta)$ and $T(\eta)$ from (20) and collecting the coefficients of M and M^2, we get

$$U_1'' = T_1 \quad \ldots(22)$$

$$U_2'' = T_2 \quad \ldots(23)$$

$$U_3'' + 3\,SU_1''\,(U_1')^2 = T_3 \quad \ldots(24)$$

$$T_1'' + \xi = 0 \quad \ldots(25)$$

$$T_2'' + U_1'^2 = 0 \quad \ldots(26)$$

$$T_3'' + 2U_1'\,U_2' = 0. \quad \ldots(27)$$

Solving eqns. (22)-(27) under the boundary conditions (21), we get

$$T_1(\eta) = -\frac{\xi\eta^2}{2} + \left(m + \frac{\xi}{2} - 1\right)\eta + 1 \quad \ldots(28)$$

$$U_1(\eta) = \frac{\xi\eta\,(1-\eta^3)}{24} + \left(m + \frac{\xi}{2} - 1\right)\frac{\eta\,(\eta^2 - 1)}{6} + \frac{\eta\,(\eta - 1)}{2} \quad \ldots(29)$$

$$T_2(\eta) = k_1(\eta - \eta^{10}) - k_2(\eta - \eta^9) + k_3(\eta - \eta^8) + k_4(\eta - \eta^7) + k_5(\eta - \eta^6) + k_6(\eta - \eta^5) + k_7(\eta - \eta^4) \quad \ldots(30)$$

$$U_2(\eta) = \frac{k_1}{132}(22\eta^3 - 21\eta - \eta^{12}) - \frac{k_2}{330}(55\eta^3 - 3\eta^{11} - 52\eta) + \frac{k_3}{90}(15\eta^3 - \eta^{10} - 14\eta) + \frac{k_4}{72}(12\eta^3 - \eta^9 - 11\eta) + \frac{k_5}{168}(28\eta^3 - 3\eta^8 - 25\eta) + \frac{k_6}{42}(7\eta^3 - \eta^7 - 6\eta) + \frac{k_7}{30}(5\eta^3 - \eta^6 - 4\eta) \quad \ldots(31)$$

where

$$\left.\begin{array}{l} k_1 = \dfrac{\xi^2}{51840},\ k_2 = \dfrac{\xi A}{5184},\ k_3 = \dfrac{2A^2 - 3\xi}{4032} \\ k_4 = \dfrac{4A\xi + 48A + 12\xi - \xi^2}{12096} \\ k_5 = \dfrac{18 + \xi A - 12A - 4A^2}{2160} \\ k_6 = \dfrac{\xi - 12 - 4A}{480},\ A = \left(m + \dfrac{\xi}{2} - 1\right) \\ k_7 = \dfrac{\xi^2 + 16A^2 + 144 - 8A\xi + 96A - 24\xi}{6912} \end{array}\right\} . \quad \ldots(32)$$

We have solved for U_3 (M) and T_3 (m) from equations (24) and (27) by using the values of U_1, T_1, U_2, T_2 and their behaviour with respect to variation in the physical parameters ξ, m and S has been shown through graphs in Figs. (3) and (6).

Discussion

Figures (1)-(8) for velocity and temperature distributions have been plotted for different values of the heat source parameter (ξ = 10, 20), the ratio of temperature of the walls (m = 0, 1, 2) and viscoelastic parameter (S = 1, 4). It is evident from Figs. 1, 2, 3 that the first and second order perturbation temperatures increase and third order perturbation temperature decrease with increase in m and is not affected by the elasticity of the fluid. First and second order perturbation velocity fields are not affected by elasticity present in the fluid which decrease with the increase in m (cf. Figs. 4, 5). Effect of elasticity is to increase the third order perturbation velocity with the increase in m as shown in Fig. 6. Both velocity and temperature fields increase with the increase in buoyancy force parameter M and heat source parameter ξ (cf. Fig. 1 through 8).

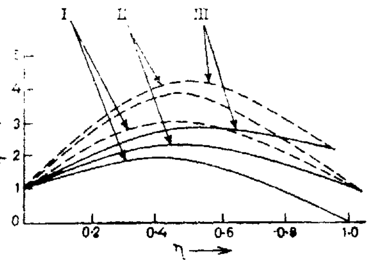

FIG. 1. Graphs for T_1 versus η. [m = 0, 1, 2, respectively for curves I, II, III. ξ = 10 for —, ξ = 20 for ⋯]

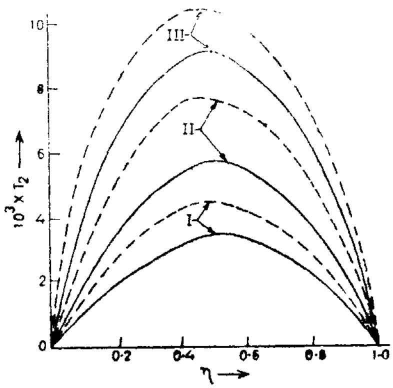

FIG. 2. Graphs for T_2 versus η. [m = 0, 1, 2, respectively for curves I, II, III, ξ = 10 for—, ξ = 20 for ⋯]

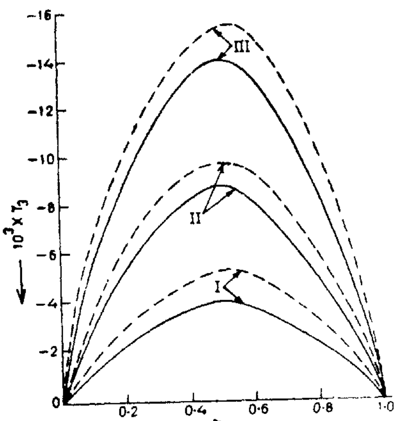

FIG. 3. Graphs for T_3 versus η. [m = 0, 1, 2, respectively for curves I, II, III, ξ = 10 for — , ξ = 20 for ⋯]

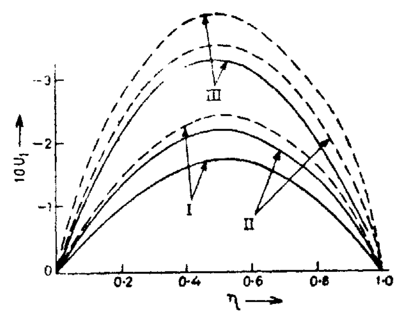

FIG. 4. Graphs for U_1 versus η. [m = 0, 1, 2, respectively for curves I, II, III, ξ = 10 for — , ξ = 20 for ⋯]

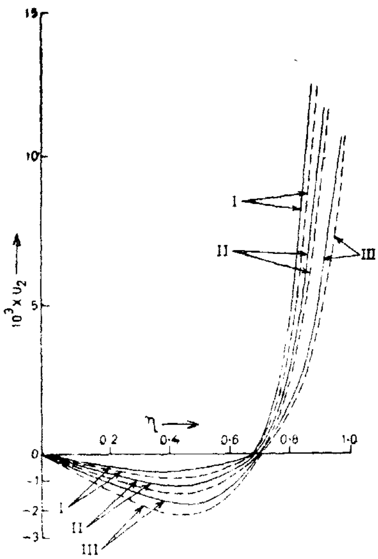

FIG. 5. Graphs for U_2 versus η. [$m = 0, 1, 2$, respectively for curves I, II, III, $\xi = 10$ for —, $\xi = 20$ for …]

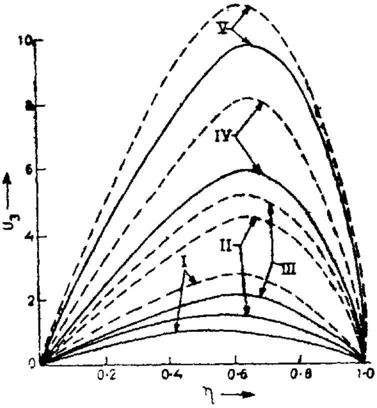

FIG. 6. Graphs for U_3 versus η.
$\xi = 10$ for —, $\xi = 20$ for …

curve	I	II	III	IV	V
S	1	1	1	4	4
M	0	1	2	1	2

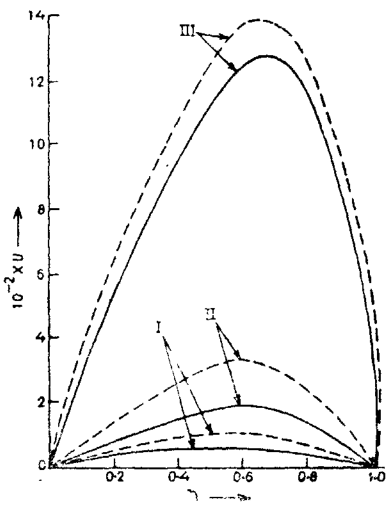

FIG. 7. Graphs for U versus η.
$\xi = 0$ for —, $\xi = 20$ for …

curve	I	II	III
m	2	2	2
S	1	1	1
M	0.1	0.2	0.4

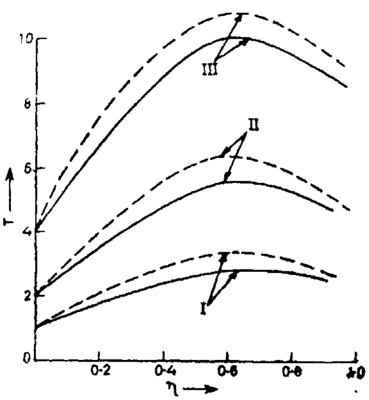

FIG. 8. Graphs for U versus η.
$\xi = 0$ for —, $\xi = 20$ for …

curve	I	II	III
m	2	2	2
S	1	1	1
M	0.1	0.2	0.4

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