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MULTIPOLE EXPANSIONS FOR TWO SUPERPOSED FLUIDS, EACH OF FINITE DEPTH

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Problems dealing with the generation of internal waves at the surface separating two fluids involves the consideration of different types of singularities in one of the two fluids. In this paper the velocity potentials describing line and point multipoles are obtained for the case when each fluid is of finite constant depth, neglecting effects of surface tension at the surface of separation.

1. Introduction

The study of surface waves in one fluid or internal waves in two fluids involves the consideration of singularities of different types in the fluids. In a previous paper (Gorgui and Kassem 1978) the velocity potentials describing these singularities were obtained, neglecting effects of surface tension at the surface of separation, for the cases when the lower fluid is of infinite and of finite constant depth while the upper being of infinite height.

In this paper we give a discussion of the basic line and point multipoles used in the two-fluids problem when each fluid is of finite constant depth and the two superposed fluids be confined between rigid horizontal planes. In the two-dimensional motion, the line singularities considered are symmetric (or vertical) multipoles. but the corresponding antisymmetric (or horizontal) multipoles can be found similarly. For axisymmetric motion, the point singularities considered are multipole singularities. These time-harmonic singularities are described by harmonic potential functions which satisfy two linearised conditions at the surface of separation, and uniqueness is ensured by requiring that there are only outgoing waves in the far field. The method used is an extension of that used in Rhodes-Robinson (1970), Thorne (1953) and Ursell (1950) for one-fluid problems, and in Gorgui and Kassem (1978) for two-fluids problems, and is valid only for submerged singularities.

2. Statement of the Problem

We are concerned with the irrotational motion of two superposed non-viscous incompressible fluids under the action of gravity, neglecting any effect due to surface tension at the surface separating the two fluids. Each fluid is of infinite horizontal extent and if we take the origin O at the mean level of the interface and the axis Oy pointing vertically downwards into the lower fluid, let the two fluids be confined between rigid horizontal planes $y = h, y = -h'$.

The motion is simple harmonic with a small amplitude and angular frequency σ; it is due to an oscillating singularity in one of the two fluids. In two-dimensional motion the singularity is a line multipole and in axisymmetric motion it is a point multipole. In each case, the velocity potentials of the lower and upper fluids are simple harmonic with period $2\pi/\sigma$ and it is more convenient to use the complex-valued potentials $\phi\, e^{-i\sigma t}$, $\phi'\, e^{-i\sigma t}$, of which the actual velocity potentials are the real parts. These potentials satisfy a boundary-value problem in which

$$\nabla^2 \phi = 0 \qquad \ldots(2.1)$$

$$\nabla^2 \phi' = 0\,. \qquad \ldots(2.2)$$

in the regions occupied by the fluids, except at the singularity;

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on } y = h \qquad \ldots(2.3)$$

$$\frac{\partial \phi'}{\partial y} = 0 \qquad \text{on } y = -h' \qquad \ldots(2.4)$$

and the linearised interface conditions

$$\left.\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{\partial \phi'}{\partial y}\,, \\ K\phi + \frac{\partial \phi}{\partial y} &= s\left(K\phi' + \frac{\partial \phi'}{\partial y}\right) \end{aligned}\right\} \text{ on } y = 0 \qquad \ldots(2.5)$$

where $K = \sigma^2/\mathbf{g}$ and s is the ratio of the density of the upper to that of the lower fluid. These conditions are applied for each singularity considered. They are supplemented by the two general limiting conditions that ϕ or ϕ' behaves like a typical singular harmonic function near the singularity, and the radiation condition that both functions represent outgoing waves in the far field.

3. Submerged Line Multipoles

Let the singularity be placed on the axis Oy at a distance f from O. We define polar coordinates (r, θ) based on the singularity position by the equations

$$x = r \sin \theta, \quad y \mp f = r \cos \theta$$

according as the singularity is in the lower or upper fluid, so that r denotes the distance from the singularity.

(i) *Multipoles Submerged in Lower Fluid*

Here ϕ and ϕ' are solutions of the boundary-value problem stated above where

$$\phi \sim \frac{\cos (n+1)\,\theta}{r^{n+1}} \text{ as } r = \left\{ x^2 + (y-f)^2 \right\}^{1/2} \to 0, \quad n=0, 1, \ldots$$

We try as solutions

$$\phi = \frac{\cos (n+1)\,\theta}{r^{n+1}} + \int_0^\infty \left\{ A(k) \cosh k\,(h-y) + B(k) \sinh ky \right\} \cos kx\, dk,$$

$$\phi' = \int_0^\infty A'(k) \cosh k(h' + y) \cos kx \, dk$$

and use the representations

$$\frac{\cos(n+1)\theta}{r^{n+1}} = \begin{cases} \dfrac{1}{n!}\displaystyle\int_0^\infty k^n e^{-k(y-f)} \cos kx \, dk, & y > f \\ \dfrac{(-1)^{n+1}}{n!}\displaystyle\int_0^\infty k^n e^{k(y-f)} \cos kx \, dk, & y < f. \end{cases}$$

It is evident that ϕ, ϕ' as given above are harmonic and that ϕ' satisfies condition (2.4). Conditions (2.3), (2.5) are satisfied if

$$B = \frac{k^n}{n!} e^{-k(h-f)} \operatorname{sech} kh$$

$$A \sinh kh + A' \sinh kh' = B + \frac{(-1)^{n+1}}{n!} k^n e^{-kf}$$

$$- KA \cosh kh + \left\{ s K \cosh kh' - (1-s) k \sinh kh' \right\} A' = \frac{(-1)^{n-1}}{n!} Kk^n e^{-kf}$$

These determine A, A', B which when substituted in the above assumed forms, give

$$\phi = \frac{\cos(n+1)\theta}{r^{n+1}} - \frac{1}{n!} \int_0^\infty \frac{k^n}{\Delta} P(n) \cos kx \, dk \qquad \text{...(3.1)}$$

$$\phi' = \frac{M}{n!\,(1+s)} \int_0^\infty \frac{k^n}{\Delta} \left\{ e^{-k(h-f)} + (-1)^{n+1} e^{k(h-f)} \right\} \cosh k(h'+y) \cos kx \, dk . \qquad \text{...(3.2)}$$

where

$$\Delta(k) = \frac{M}{1+s} (\cosh kh \sinh kh' + s \sinh kh \cosh kh') - k \sinh kh \sinh kh'$$

$$= \frac{1}{4}(k+K) e^{-k(h-h')} + \frac{1}{4}(k-K) e^{k(h-h')}$$

$$- \frac{1}{4}(k-M) e^{k(h+h')} - \frac{1}{4}(k+M) e^{-k(h+h')} \qquad \text{...(3.3)}$$

$$M = (1+s) K/(1-s)$$

and

$$P(n) = \Bigg[(-1)^{n+1} \left\{ k \sinh kh' + \frac{M}{1+s} (\sinh kh' - s \cosh kh') \right\} e^{-kf}$$

$$\times \cosh k(h-y) + e^{-k(h-f)} \left\{ k \sinh kh' \cosh ky - \frac{M}{1+s} \right.$$

$$\left. \times (\sinh kh' \sinh ky + s \cosh kh' \cosh ky) \right\} \Bigg]. \qquad \text{...(3.4)}$$

Now, $\Delta(k)$ has one simple zero at $k = m$, say, on the real axis of k. This introduces simple poles for the integrals in ϕ, ϕ'. Below this pole we make an indentation of the contours of integrations in (3.1), (3.2). By putting $2\cos kx = e^{ik|x|} + e^{-ik|x|}$, and rotating the contours in the indented integrals in ϕ, ϕ' into contours in the first and fourth quadrants so that we must include the residue term at $k=m$ for the first, we obtain the outgoing waves

$$\phi \sim (n+1)\, C(n+1)\, \frac{\cosh m(h-y)}{m\sinh mh}\, e^{im|x|}$$

$$\phi' \sim -(n+1)\, C(n+1)\, \frac{\cosh m(h'+y)}{m\sinh mh'}\, e^{im|x|}$$

as $|x| \to \infty$, where

$$C(n) = -\frac{2\pi i M m^n}{n!\,(1+s)}\,(1-e^{-2mh'})\,(e^{-m(h-f)} + (-1)^n e^{m(h-f)})$$
$$\times \Big[e^{-mh} + (2hM-2hm-1)\, e^{mh} + (2h'M+2h'm-1)\, e^{-m(h+2h')}$$
$$+ (2(h-h')(m-K)+1)\, e^{m(h-2h')} \Big]^{-1} . \qquad \ldots(3.5)$$

(ii) *Multipoles Submerged in Upper Fluid*

The boundary-value problem for ϕ, ϕ' in this case is given by eqns. (2.1) to (2.5), supplemented by the condition

$$\phi \sim \frac{\cos(n+1)\theta}{r^{n+1}} \text{ as } r = \left\{ x^2 + (y+f)^2 \right\}^{1/2} \to 0, \quad n = 0,1,\ldots.$$

We try as solutions

$$\phi = \int_0^\infty A(k) \cosh k(h-y) \cos kx \, dk$$

$$\phi' = \frac{\cos(n+1)\theta}{r^{n+1}} + \int_0^\infty \left\{ A'(k) \cosh k(h'+y) + B'(k) \sinh ky \right\} \cos kx \, dk$$

where ϕ satisfies condition (2.3). Conditions (2.4), (2.5) are satisfied if

$$B' = \frac{(-1)^n}{n!} k^n e^{-k(h'-f)} \operatorname{sech} kh'$$

$$A \sinh kh + A' \sinh kh' = -B' + \frac{k^n}{n!} e^{-kf}$$

$$\left\{ K\cosh kh - (1-s)\, k \sinh kh \right\} A - sK A' \cosh kh' = \frac{sK}{n!} k^n e^{-kf}.$$

These lead to

$$\phi = -\frac{s\,M}{n!\,(1+s)} \int_0^\infty \frac{k^n}{\Delta} \left\{ e^{k(h'-f)} + (-1)^{n+1} e^{-k(h'-f)} \right\} \cosh k(h-y) \cos kx \, dk$$
$$\ldots(3.6)$$

$$\phi' = \frac{\cos(n+1)\,\theta}{r^{n+1}} - \frac{1}{n!}\int_0^\infty \frac{k^n}{\Delta} Q(n) \cos kx \, dk \qquad \ldots(3.7)$$

where Δ is given by eqn. (3.3) and

$$Q(n) = \Big[\Big\{k\sinh kh - \frac{M}{1+s}(\cosh kh - s\sinh kh)\Big\} e^{-kf}\cosh k(h'+y)$$
$$+ (-1)^{n+1} e^{-k(h'-f)}\Big\{ k\sinh kh\cosh ky - \frac{M}{1+s}(\cosh kh\cosh ky$$
$$- s\sinh kh\sinh ky)\Big\}\Big]. \qquad \ldots(3.8)$$

The contours of the integrals in (3.6), (3.7) are indented below the simple pole at $k=m$ to give the outgoing waves

$$\phi \sim (n+1)\, C'(n+1) \frac{\cosh m(h-y)}{m\sinh mh} e^{im|x|}$$

$$\phi' \sim -(n+1)\, C'(n+1) \frac{\cosh m(h'+y)}{m\sinh mh'} e^{im|x|}$$

as $|x| \to \infty$, where

$$C'(n) = \frac{4\pi\, is\, Mm^n}{n!\,(1+s)}(e^{-mf} + (-1)^n e^{-m(2h'-f)}) \sinh mh$$
$$\times\Big[e^{-mh} + (2hM - 2hm - 1)\, e^{mh} + (2h'M + 2h'm - 1)\, e^{-m(h-2h')}$$
$$+ (2(h-h')(m-K) + 1)\, e^{m(h-2h')}\Big]^{-1}. \qquad \ldots(3.9)$$

4. Submerged Point Multipoles

We now define cylindrical polar coordinates (r, ψ, y), where r is the distance from the y-axis, and also spherical polar coordinates (R, θ, ψ) based on the singularity position, by the equations

$$r = R\sin\theta, \qquad y \mp f = \cos\theta.$$

R therefore denotes the distance from the singularity. We consider only point singularities for which Oy is an axis of symmetry, so that the velocity potentials ϕ, ϕ' are independent of the angle ψ.

The boundary-value problem for ϕ, ϕ' in this case is given by eqns. (2.1) to (2.5), supplemented by the two limiting conditions near the singularity and in the far field.

(i) *Multipoles Submerged in Lower Fluid*

Here

$$\phi \sim \frac{P_n(\cos\theta)}{R^{n+1}} \text{ as } R = \Big\{ r^2 + (y-f)^2\Big\}^{1/2} \to 0,\ n=0, 1, \ldots . \qquad \ldots(4.1)$$

If we try as solutions

$$\phi = \frac{P_n(\cos\theta)}{R^{n+1}} + \int_0^\infty \left\{ A(k)\cosh k(h-y) + B(k)\sinh ky \right\} J_0(kr)\, dk$$

$$\phi' = \int_0^\infty A'(k)\cosh k(h'+y)\, J_0(kr)\, dk$$

then using the representations

$$\frac{P_n(\cos\theta)}{R^{n+1}} = \begin{cases} \dfrac{1}{n!}\displaystyle\int_0^\infty k^n e^{-k(y-f)} J_0(kr)\, dk, & y > f \\ \dfrac{(-1)^n}{n!}\displaystyle\int_0^\infty k^n e^{k(y-f)} J_0(kr)\, dk, & y < f \end{cases}$$

conditions (2.3), (2.5) are satisfied if

$$B = \frac{k^n}{n!} e^{-k(h-f)} \operatorname{sech} kh$$

$$A \sinh kh + A' \sinh kh' = B + \frac{(-1)^n}{n!} k^n e^{-kf}$$

$$-KA\cosh kh + \left\{ sK\cosh kh' - (1-s)k\sinh kh' \right\} A' = \frac{(-1)^n}{n!} K k^n e^{-kf}.$$

Solving these equations and substituting, we obtain

$$\phi = \frac{P_n(\cos\theta)}{R^{n+1}} - \frac{1}{n!} \int_0^\infty \frac{k^n}{\Delta} P(n-1)\, J_0(kr)\, dk \qquad \ldots(4.2)$$

$$\phi' = \frac{M}{n!(1+s)} \int_0^\infty \frac{k^n}{\Delta} \left\{ e^{-k(h-f)} + (-1)^n e^{k(h-f)} \right\} \cosh k(h'+y)\, J_0(kr)\, dk \qquad \ldots(4.3)$$

where Δ is given by (3.3), $P(n)$ is given by (3.4) and, as before, the contour of integration is indented below the simple root $k = m$ of $\Delta = 0$ on the positive real k-axis, which ensures that the radiation conditions are satisfied. For, by putting

$$2J_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr)$$

rotating the contours in each integral into contours in the first and fourth quadrants (where $H_0^{(1)}(kr)$, $H_0^{(2)}(kr)$ are respectively small) and including the residue term at $k = m$ for the first, we obtain the diverging cylindrical waves

$$\phi \sim C(n) \frac{\cosh m(h-y)}{\sinh mh} H_0^{(1)}(mr)$$

$$\phi' \sim -C(n) \frac{\cosh m(h'+y)}{\sinh mh'} H_0^{(1)}(mr)$$

as $r \to \infty$, where $C(n)$ is given by eqn. (3.5).

(ii) *Multipoles Submerged in Upper Fluid*

In this case condition (4.1) is replaced by

$$\phi' \sim \frac{P_n(\cos\theta)}{R^{n+1}} \text{ as } R = \left\{ r^2 + (y+f)^2 \right\}^{1/2} \to 0, \; n=0, 1, \ldots \qquad \ldots(4.4)$$

Try as solutions the harmonic potentials

$$\phi = \int_0^\infty A(k) \cosh k(h-y) J_0(kr)\, dk$$

$$\phi' = \frac{P_n(\cos\theta)}{R^{n+1}} + \int_0^\infty \left\{ A'(k) \cosh k(h'+y) + B'(k) \sinh ky \right\} J_0(kr)\, dk$$

where ϕ satisfies condition (2.3). Conditions (2.4), (2.5) are satisfied if

$$B' = \frac{(-1)^{n+1}}{n!} k^n e^{-k(h'-f)} \operatorname{sech} kh',$$

$$A \sinh kh + A' \sinh kh' = -B' + \frac{k^n}{n!} e^{-kf},$$

$$\left\{ K \cosh kh - (1-s) k \sinh kh \right\} A - s K A' \cosh kh' = \frac{sK}{n!} k^n e^{-kf}.$$

These lead to

$$\phi = \frac{sM}{n!(1+s)} \oint_0^\infty \frac{k^n}{\Delta} \left\{ e^{k(h'-f)} + (-1)^n e^{-k(h'-f)} \right\} \cosh k(h-y) J_0(kr)\, dk \qquad \ldots(4.5)$$

$$\phi' = \frac{P_n(\cos\theta)}{R^{n+1}} - \frac{1}{n!} \oint_0^\infty \frac{k^n}{\Delta} Q(n-1) J_0(kr)\, dk \qquad \ldots(4.6)$$

where Δ is given by (3.3), $Q(n)$ is given by (3.8) and the path of integration is indented below the simple pole at $k=m$ to give the diverging cylindrical waves

$$\phi \sim C'(n) \frac{\cosh m(h-y)}{\sinh mh} H_0^{(1)}(mr)$$

$$\phi' \sim -C'(n) \frac{\cosh m(h'+y)}{\sinh mh'} H_0^{(1)}(mr)$$

as $r \to \infty$, where $C'(n)$ is given by eqn. (3.9).

5. Conclusion

Known results in the absence of the upper fluid can be made evident by putting $s=0$ in the suitable formulae for ϕ. Also the known results when the upper fluid be of infinite height can be made evident by letting $h' \to \infty$ in both ϕ and ϕ'. Finally, the special case when each fluid is of infinite depth may also be determined by letting $h, h' \to \infty$ in both ϕ and ϕ'.

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