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A NOTE ON SEMI-CONTINUITY AND PRECONTINUITY

A. S. MASHHOUR, I. A. HASANEIN AND S. N. EL-DEEB

Department of Mathematics, Faculty of Science, University of Assiut, Assiut, Egypt

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Noiri (1973 a,b) has investigated some properties of semi-continuous mappings. Mashhour *et al.* (1980) introduced and studied the concepts of precontinuous and preopen mappings. In this paper we succeed to strengthen some results of Mashhour *et al.* (1980) and Noiri (1973 a,b) and to introduce more properties of precontinuous mappings.

INTRODUCTION

Let X, Y and Z be topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp. interior) of S will be denoted by $\bar{S}$ (resp. S^0). A subset S of X is called semi-open (Levine 1963) [resp. preopen (Mashhour *et al.* 1980)] if $S \subset S^{0-}$ (resp. $S \subset S^{-0}$). The complement of a semi-open (resp. preopen) set is called semi-closed (Biswas 1970) [resp. preclosed (Mashhour *et al.* 1980)]. The family of all semi-open (resp. preopen) sets will be denoted by $SO(X)$ (resp. $PO(X)$). It is clear that, openness implies preopenness and semi-openness, but the converses are not true.

Levine (1963) has defined a mapping $f : X \to Y$ to be semi-continuous (briefly s.c.) if the inverse of each open set in Y is semi-open in X. Noiri (1973a) has defined a mapping $f : X \to Y$ to be semi-closed if the image of each closed set in X is semi-closed.

Noiri (1973a,b) has obtained an extensive list of theorems and corollaries about semi-continuity, and among them, the following are established :

Theorem A (Noiri 1973a)— If $f : X \to Y$ is an open s.c. mapping, then $f^{-1}(B) \in SO(X)$ for each $B \in SO(Y)$.

Corollary B (Noiri 1973a)— If $f : X \to Y$ is an open s.c. mapping, then the inverse of each semi-closed set in Y is semi-closed in X.

Corollary C (Noiri 1973b)— If $f : X \to Y$ is open s.c. and $g : Y \to Z$ is s.c., then $g \circ f : X \to Z$ is s.c.

Theorem D (Noiri 1973a)— Let $f : X \to Y$ and $g : Y \to Z$ be two mappings and $g \circ f : X \to Z$ be a semi-closed mapping. Then, f is semi-closed if g is an injective, s.c., open mapping.

Theorem E (Noiri 1973b)— If $f : X \to Y$ is s.c. and $X_0 \subset X$ is open, then the restriction $f/X_0 : X_0 \to Y$ is s.c.

Mashhour *et al.* (1980) have defined a mapping $f : X \to Y$ to be precontinuous, briefly p.c. (resp. preopen) if the inverse image (resp. image) of each open set in Y

(resp. X) is preopen in X (resp. Y). They obtained in Mashhour *et al.* (1980) several results containing the following theorems and lemmas:

Theorem F— If $f : X \to Y$ is p.c. and $U \subset X$ is open, then the restriction $f/U : U \to Y$ is p.c.

Lemma G— If $U \subset X$ is open and $V \in PO(X)$, then $U \cap V \in PO(X)$.

Lemma H— If $U \subset X$ is open and $V \in PO(U)$, then $V \in PO(X)$.

Theorem J— Let $f : X \to Y$ be a mapping and $\{U_i : i \in I\}$ be an open cover of X. Then f is p.c. if f/U_i is p.c. for each $i \in I$.

In the present note, the above results are strengthened and more properties of precontinuity are established.

1. Semi-Continuity

Now, we show that the word 'open mapping' in Theorem A, Corollaries B and C and Theorem D can be replaced by "preopen mapping" and the word "open set" in Theorem E can be replaced by "preopen set".

Theorem 1.1—If $f : X \to Y$ is a preopen s.c. mapping then, $f^{-1}(B) \in SO(X)$ for each $B \in SO(Y)$.

Proof : Suppose $B \in SO(Y)$, then there exists an open set V in Y such that $V \subset B \subset \overline{V}$. So, $f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(\overline{V}) \subset (f^{-1}(V))^-$ (see Mashhour *et al.* 1980). Since f is s.c., then $f^{-1}(V) \in SO(X)$. Then, there exists an open set $G \subset X$ such that $G \subset f^{-1}(V) \subset f^{-1}(B) \subset (f^{-1}(V))^- \subset \overline{G}$. Therefore, $f^{-1}(B) \in SO(X)$.

Corollary 1.1—If $f : X \to Y$ is preopen and s.c., then the inverse image of each semi-closed set in Y is semi-closed in X.

Proof : Let $B \subset Y$ be a semi-closed set. By Theorem 1.1, $f^{-1}(Y-B) \in SO(X)$. But, $f^{-1}(Y-B) = X - f^{-1}(B)$, so $f^{-1}(B)$ is semi-closed in X.

Corollary 1.2—If $f : X \to Y$ is preopen s.c. and $g : Y \to Z$ is s.c., then $g \circ f$ is s.c.

Proof : This is an immediate consequence of Theorem 1.1.

Theorem 1.2—Let $f : X \to Y$ and $g : Y \to Z$ be two mappings and let $g \circ f : X \to Z$ is semi-closed. Then, f is semi-closed if g is an injective preopen s.c. mapping.

Proof : Let FX be closed, so $g \circ f(F)$ is semi-closed in Z. Since g is injective we have $g^{-1}((g \circ f)(F)) = f(F)$. It follows immediately from Corollary 1.1 that $f(F)$ is semi-closed, because g is preopen s.c. Hence, f is semi-closed.

The following lemma is very useful in the sequel.

Lemma 1.1—If $U \in PO(X)$ and $V \in SO(X)$, then $U \cap V \in SO(U)$.

Proof : Since $U \cap V \subset U^{-0} \cap V^{0-} \subset (U^{-0} \cap V^0)^- \subset (U^- \cap V^0)^- \subset (U \cap V^0)^-$. So, $U \cap V \subset (U \cap V^0)^- \cap U = \mathrm{Cl}_U (U \cap V^0)$. Since V^0 is open in X, hence $U \cap V^0$ is open in U. Thus, $U \cap V \subset \mathrm{Cl}_U (\mathrm{Int}_U (U \cap V^0)) \subset \mathrm{Cl}_U (\mathrm{Int}_U (U \cap V))$, where $\mathrm{Cl}_U(\ )$ and $\mathrm{Int}_U(\ )$ denote the closure and the interior, respectively, in the subspace U. Therefore, $U \cap V \in SO(U)$.

Theorem 1.3—If $f : X \to Y$ is s.c. and $X_0 \in PO(X)$, then the restriction $f/X_0 : X_0 \to Y$ is s.c.

PROOF : Let $V \subset Y$ be an open set. Then, $f^{-1}(V) \in SO(X)$. Since $X_0 \in PO(X)$, by Lemma 1.1, $f^{-1}(V) \cap X_0 = (f/X_0)^{-1}(V) \in SO(X_0)$. Hence, f/X_0 is s.c.

2. PRE-CONTINUITY

Now, we show that "open set" in Lemma G and Theorem F (resp. Lemma H and Theorem J) can be replaced by "semi-open (resp. preopen) set".

Lemma 2.1—If $V \in PO(X)$ and $U \in SO(X)$, then $U \cap V \in PO(U)$.

PROOF : Since $U \cap V \subset U \cap V^{-0} = \text{Int}_U (U \cap V^{-0}) \subset \text{Int}_U (U^{0-} \cap V^{-0}) \subset \text{Int}_U ((U^0 \cap V^{-0})^-) \subset \text{Int}_U ((U^0 \cap V^-)^-) \subset \text{Int}_U ((U \cap V)^-)$. So, $U \cap V \subset \text{Int}_U ((U \cap V)^-) \cap U = \text{Int}_U ((U \cap V)^- \cap U) = \text{Int}_U ((\text{Cl}_U (U \cap V))$. This implies that $U \cap V \in PO(U)$.

Theorem 2.1—If $f : X \to Y$ is p.c. and $U \in SO(X)$, then the restriction $f/U : U \to Y$ is p.c.

PROOF : Let $V \subset Y$ be an open set, so $f^{-1}(V) \in PO(X)$. Since $U \in SO(X)$, by Lemma 2.1, we have $f^{-1}(V) \cap U = (f/U)^{-1}(V) \in PO(U)$. Therefore, f/U is p.c.

Lemma 2.2—If $U \in PO(X)$, and $V \in PO(U)$ then $V \in PO(X)$.

PROOF : Since $V \subset \text{Int}_U(\text{Cl}_U(V))$ and $\text{Int}_U(\text{Cl}_U(V))$ is open in U, there exists an open set $W \subset X$ such that $U \cap W = \text{Int}_U(\text{Cl}_U(V))$, so $V \subset U^{-0} \cap W \subset (U \cap W)^{-0} = (\text{Int}_U(\text{Cl}_U(V)))^{-0} \subset (\text{Cl}_U(V))^{-0} \subset (V^-)^{-0} = V^{-0}$. Then, $V \in PO(X)$.

Theorem 2.2—Let $f : X \to Y$ be a mapping and $\{U_i : i \in I\}$ a cover of X such that $U_i \in PO(X)$ for each $i \in I$. Then, f is p.c. if f/U_i is p.c. for each $i \in I$.

PROOF : Let $V \subset Y$ be an open set, so $(f/U_i)^{-1}(V) = f^{-1}(V) \cap U_i \in PO(U_i)$. Since $U_i \in PO(X)$, by Lemma 2.2, $(f/U_i)^{-1}(V) \in PO(X)$ for each $i \in I$. But, $f^{-1}(V) = \bigcup_{i \in I} (f/U_i)^{-1}(V)$, so $f^{-1}(V) \in PO(X)$. This shows that f is p.c.

In what follows, we introduce new properties of precontinuity.

Theorem 2.3—If $f : X \to Y$ is p.c. and Y is a Hausdorff space, then the graph $G(f) = \{(x, f(x)) : x \in X\}$ of f is preclosed.

PROOF : Let $(x, y) \notin G(f)$, so $y \neq f(x)$. Since Y is Hausdorff space, there exist two disjoint open sets V and W such that $f(x) \in W$ and $y \in V$. Since f is p.c., there exists $U \in PO(X)$ such that $x \in U$ and $f(U) \subset W$ (Mashhour *et al.* 1980, Theorem 1). Therefore, we obtain $(x, y) \in U \times V \subset X \times Y - G(f)$. Since V is open, then $U \times V \in PO(X \times Y)$ (see Mashhour *et al.* 1980) and so $X \times Y - G(f)$ is preopen. Then $G(f)$ is preclosed in $X \times Y$.

Theorem 2.4—If $f : X \to Y$ is p.c. and Y is a Hausdorff space, then the set $\{(x_1, x_2) : f(x_1) = f(x_2)\}$ preclosed in the product space $X \times X$.

PROOF : Let Δ be the diagonal of $Y \times Y$. Since Y is Hausdorff space, so Δ is a closed set of $Y \times Y$, as well known. Since f is p.c., $f \times f : X \times X \to Y \times Y$ is so (see Mashhour *et al.* 1980). Therefore, $(f \times f)^{-1}(\Delta)$ is preclosed [Mashhour *et al.* 1980, Theorem 1]. But, $(f \times f)^{-1}(\Delta) = \{(x_1, x_2) : f(x_1) = f(x_2)\}$. The proof is completed.

Definition 2.1 (Noiri *et al.* 1980) – Let X be a topological space and A a subset of X. The set $\cap \{F \subset X : A \subset F$ and F is preclosed in $X\}$ is called the preclosure of A and is denoted by Pcl (A).

Lemma 2.3 (Noiri *et al.* 1980)—A set $S \subset X$ is preclosed if and only if $\text{Pcl}(S) = S$.

Definition 2.2—Let $A \subset X$ be a set. A mapping $f : X \to A$ is called a precontinuous retraction (briefly p.c. ret.) if f is p.c. and f/A is identity.

Theorem 2.5—Let $A \subset X$ be a set and $f : X \to A$ be a p.c. ret. mapping. If X is a Hausdorff space, then A is a preclosed set of X.

PROOF : Suppose that A is not preclosed. By Lemma 2.3, there exists a point $x \in X$ such that $x \in \text{Pcl}(A)$ but $x \notin A$. So, $f(x) \neq x$. Since X is a Hausdorff space, there exist disjoint open sets U and V in X such that $x \in U$, $f(x) \in V$. Now, let W be an arbitrary preopen set containing x. Then, $x \in W \cap U \in PO(X)$. Since x Pcl (A), by Definition 2.1, $(W \cap U) \cap A \neq \phi$. So, there exists a point $y \in (W \cap U) \cap A$. Since $y \in A$, $f(y) = y$ and hence $f(y) \notin V$. Therefore $f(W) \not\subset V$. This is contrary to the precontinuity of f. Consequently, A is a preclosed set set of X.

Lemma 2.4 (Noiri *et al.* 1980)—Let $\{X_i : i \in I\}$ be a family of topological spaces, $X = \pi X_i$ the product space and $A = \prod_{j=1}^{n} A_{i_j} \times \prod_{i \neq i_j} X_i$ a non-empty subset of X, where n is a positive integer and A_{i_j} is a subset of X_{i_j}. Then, $A_{i_j} \in PO(X_{i_j})$ for each j $(1 \leqslant j \leqslant n)$ if and only if $A \in PO(X)$.

Theorem 2.6—Let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be any two families of topological spaces. For each $i \in I$, let $f_i : X_i \to Y_i$ be a mapping. Then, a mapping $f : \Pi X_i \to \Pi Y_i$ defined by $f((x_i)) = (f_i((x_i))$ is p.c. if and only if f_i is p.c. for each $i \in I$.

PROOF : *Necessity*—For each fixed $i_0 \in I$, let $P_{i_0} : \Pi Y_i \to Y_{i_0}$ be the projection. Suppose V_{i_0} is an arbitrary open set in Y_{i_0}. Then, $P_{i_0}^{-1}(V_{i_0}) = V_{i_0} \times \prod_{i \neq i_0} Y_i$ is open in ΠY_i. Since f is p.c., $f^{-1}(P_{i_0}^{-1}(V_{i_0})) = f_{i_0}^{-1}(V_{i_0}) \times \prod_{i \neq i_0} X_i$ is preopen in ΠX_i, Then by Lemma 2.4, $f_{i_0}^{-1}(V_{i_0}) \in PO(X_{i_0})$. Therefore, f_{i_0} is p.c.

Sufficiency—Suppose V is a basic open set of ΠY_i. Then, there are $i_j \in I$ $(i \leqslant j \leqslant n)$ and open sets $V_{i_j} \subset Y_{i_j}$ such that $V = \prod_{j=1} V_{i_j} \times \prod_{i \neq i_j} Y_i$. Since f_{i_j} is p.c., $f_{i_j}^{-1}(V_{i_j}) \in PO(X_{i_j})$ for each j $(i \leqslant j \leqslant n)$. If $f_{i_j}^{-1}(V_{i_j}) = \phi$, then $f^{-1}(V)$

$= \prod_{j=1}^{n} f_{i_j}^{-1} (V_{i_j}) \times \prod_{i \neq j} X_i = \phi$. Hence, $f^{-1}(V) \in PO(\Pi X_i)$. If $f^{-1}(V_{i_j}) \neq \phi$ for all j, then $\prod_{j=1}^{n} f_{i_j}^{-1} (V_{i_j}) \times \prod_{i \neq i_j} X_i \neq \phi$. Then, by Lemma 2.4, $f^{-1}(V) \in PO(\Pi X_i)$. Now for any open set $W \subset \Pi Y_i$, there exists a family $\{V_m : m \in I\}$ of basic open sets such that $W = \bigcup_{m \in I} V_m$. Therefore, $f^{-1}(W) = \bigcup_{m \in I} f^{-1}(V_m) \in PO(\Pi X_i)$. This implies that f is p.c.

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