Indian J. pure appl. Math., 13(10) : 1124–1131, October 1982

OPERATIONAL GENERATING RELATIONS FOR CERTAIN FUNCTIONS OF SEVERAL VARIABLES

A. K. AGARWAL* AND H. L. MANOCHA

Department of Mathematics, Indian Institute of Technology, New Delhi 110016

(*Received* 3 *August* 1980; *after revision* 10 *September* 1981)

This paper is devoted to a multivariable extension of the operator ${}_aT_b = z^a \left(b + z \frac{d}{dz} \right)$ used by Joshi and Prajapat (1974, 1975, 1977), and Mittal (1977). This leads to, among others, the generalizations of the basic results of these and other authors.

1. INTRODUCTION

Following Karlsson (1975) and Srivastava (1977), let

$$F(\lambda\ ; z_1, \ldots, z_r) = \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)_M\, A(k_1, \ldots, k_r) \times z_1^{k_1} \ldots z_r^{k_r} \qquad \ldots(1.1)$$

where r is a positive integer, $(\lambda)_k$ is the Pochhammer's symbol defined by

$$(\lambda)_k = \begin{cases} 1, \text{ if } k = 0, \\ \lambda(\lambda+1)\ldots(\lambda+k-1), \text{ for all } k \in \{1,2,3, \ldots\} \end{cases} \qquad \ldots(1.2)$$

$\{A(k_1, \ldots, k_r)\}$ is a multiple complex sequence, and for convenience

$M = m_1 k_1 + \ldots + m_r k_r$,

$m_1, \ldots, m_r$ non-negative integers.

Further, assume that the series involved in (1.1) is differentiable term-by-term (any number of times) with respect to each z_i, where $i \in \{1,2,\ldots,r\}$.

Obviously, for $r = 1$ and each $m_i = 0$, the right-hand side of (1.1) represents a formal power series in z_1. Replacing z_1 by z, we shall denote this particular case by $f(z)$.

In this paper we give a multivariable extension of the operator ${}_aT_b = z^a \left(b + z \frac{d}{dz} \right)$ used by Joshi and Prajapat (1974, 1975, 1977), and Mittal (1977) which provides us with the generalizations of the basic results of these and other authors. The operational generating formulas derived in this paper may be used to obtain the corresponding results for the Lauricella functions and the Kampé de

---
*Present address : Department of Mathematics, RBS College, Agra 282002.

Fériet functions. In the last section we give a characterisation of the function F defined by (1.1).

2. Extension of ${}_aT_b$

Let

$$\underset{c_1,\ldots,c_r}{\overset{a_1,\ldots,a_r}{U_b}} = \left(\prod_{i=1}^{r} z_i^{a_i}\right)\left(b + \sum_{j=1}^{r} c_j z_j \frac{\partial}{\partial z_j}\right) \qquad \ldots(2.1)$$

where r is a positive integer and the a's, b and c's are suitable constants.
It is easily verified that

$$\underset{c_1,\ldots,c_r}{\overset{a_1,\ldots,a_r}{U_b^n}} \left(\prod_m z_m^{\beta_m}\right) = \left\{\left(\prod_i z_i^{a_i}\right)\left(\sum_j c_j a_j\right)\right\}^n \left(\frac{b+\sum_p c_p\beta_p}{\sum_l c_l a_l}\right)_n \left(\prod_m z_m^{\beta_m}\right) \qquad \ldots(2.2)$$

where (and in what follows) the limits of i, j, p, l and m are from 1 to r and $n = 0, 1, 2, \ldots,$

Using property (2.2) of the operator $\underset{c_1,\ldots,c_r}{\overset{a_1,\ldots,a_r}{U_b}}$, we propose here to prove the following :

Theorem 2.1—Let the function $F(\lambda\,;\, z_1, \ldots, z_r)$ be defined by (1.1) and the operator $\underset{c_1,\ldots,c_r}{\overset{a_1,\ldots,a_r}{U_b}}$ by (2.1). Then for arbitrary d and β_m,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \underset{c_1,\ldots,c_r}{\overset{a_1,\ldots,a_r}{U_{b+dn}^n}} \left\{\left(\prod_m z_m^{\beta_m}\right) F(\lambda\,;\, z_1, \ldots, z_r)\right\}$$

$$= \frac{\left(\prod_m z_m^{\beta_m}\right)\left(1+v\right)^{\{b+\sum_p c_p\beta_p\}/\{\sum_l c_l a_l\}}}{1-(dv/\sum_l c_l a_l)}$$

$$\times\, F\left(\lambda\,;\, z_1(1+v)^{c_1/\sum_l c_l a_l}, \ldots, z_r(1+v)^{c_r/\sum_l c_l a_l}\right) \qquad \ldots(2.3)$$

where $v = t\left(\prod_i z_i^{a_i}\right)\left(\sum_j c_j a_j\right)\left(1+v\right)^{(d/\sum_l c_l a_l)+1}$.

Proof : Starting with the left-hand side of (2.3), we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, {}^{a_1, \ldots, a_r}_{c_1, \ldots, c_r} U^n_{b+dn} \left\{ \left(\prod_m z_m^{\beta_m} \right) F\left(\lambda ; z_1, \ldots, z_r \right) \right\}$$

$$= \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)_M A(k_1, \ldots, k_r)$$

$$\times \sum_{n=0}^{\infty} \frac{t^n}{n!} \, {}^{a_1, \ldots, a_r}_{c_1, \ldots, c_r} U^n_{b+dn} \left(\prod_m z_m^{\beta_m + k_m} \right)$$

$$= \left(\prod_m z_m^{\beta_m} \right) \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)_M A(k_1, \ldots, k_r) z_1^{k_1} \ldots z_r^{k_r}$$

$$\times \sum_{n=0}^{\infty} \frac{\left\{ t \prod_i \left(z_i^{a_i} \right) \left(\sum_j c_j a_j \right) \right\}^n}{n!} \binom{\dfrac{b + dn + \sum_p c_p (\beta_p + k_p)}{\sum_l c_l a_l}}{n}$$

(by ...(2.2)

$$= \left(\prod_m z_m^{\beta_m} \right) \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)_M A(k_1, \ldots, k_r) z_1^{k_1} \ldots z_r^{k_r}$$

$$\times \sum_{n=0}^{\infty} \binom{\dfrac{b + \sum_p c_p (\beta_p + k_p) - \sum_l c_l a_l}{\sum_l c_l a_l} + \left(\dfrac{d}{\sum_l c_l a_l} + 1 \right) n}{n}$$

$$\times \left\{ \left(\prod_i^n z^{i^{a_i}} \right) \left(\sum_j c_j a_j \right) t \right\}^n$$

summing the inner series with the help of the identity [Pólya and Szegö 1925, p.126, Ex. 212]

$$\frac{(1+v)^{a+1}}{1 - bv} = \sum_{n=0}^{\infty} \binom{a + (b+1) n}{n} t^n \qquad \ldots(2.4)$$

where $v = t(1 + v)^{b+1}$, and then interpreting the resulting expression by means of (1.1), we are led immediately to (2.3). Putting $r = 1 = c_1$, $m_1 = 0 = \beta_1$, $a_1 = a$, we obtain from our Theorem 2.1 the following formula given, for example by Mittal [1977, p.24, eqn. (2.1)] (with $k = b$, $m = d$ and $x = z$) :

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, {}_aT^n_{b+dn} \{ f(z) \} = \frac{(1+v)^{b/a}}{1 - dv/a} f(z(1+v)^{1/a}) \qquad \ldots(2.5)$$

where $v = atz^a (1 + v)^{(d+a)/a}$.

For $r = 1 = a_1 = c_1$, $m_1 = 0$ and $\beta_1 = \beta$, Theorem 2.1 reduces to the following formula [Mittal 1972, p.74, eqn. (2.4)] (with $m-1 = d$, $b = \beta$, $a+1 = b$ and $x = z$):

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{b+dn} \{z^{\beta} f(z)\}$$

$$= \frac{z^{\beta}(1+v)^{b+\beta}}{1-dv} f(z(1+v)), \; v = zt(1+v)^{d+1} \qquad \ldots(2.6)$$

here, as well as throughout this paper

$$T_b \equiv {}_1T_b = z\left(b + \frac{zd}{dz}\right).$$

For $d = 0 = \beta_m$, for all $m \in \{1,\ldots,r\}$ Theorem 2.1 reduces to the formula

$$\exp\left(t \; {}^{a_1,\ldots,a_r}_{c_1,\ldots,c_r} U_b\right) \quad F(\lambda\,;\, z_1, \ldots, z_r)$$

$$= (1+v)^{b/\Sigma_l c_l a_l} \quad F\left(\lambda\,;\, z_1(1+v)^{c_1/\Sigma_l c_l a_l}, \ldots, z_r(1+v)^{c_r/\Sigma_l c_l a_l}\right) \qquad \ldots(2.7)$$

where $v = t\left(\prod_i z_i^{a_i}\right)\left(\sum_j c_j a_j\right)\left\{1 - t\left(\prod_i z_i^{a_i}\right)\left(\sum_j c_j a_j\right)\right\}^{-1}$.

Putting $r = 1 = c_1$, $m_1 = 0$ and $a_1 = a$ in (2.7), we get the following result given, for example, by Mittal [1977, p. 24, eqn. (2.3)] (with $k = b$ and $x = z$):

$$\exp(t\, {}_aT_b)\{f(z)\} = (1 - atz^a)^{-b/a} f(z(1 - atz^a)^{-1/a}). \qquad \ldots(2.8)$$

For $r = 1 = c_1$, $a_1 = -1$ and $b = 0 = m_1$, our formula (2.7) yields Taylor's theorem

$$e^{tD}\{f(z)\} = f(z + t), \; D \equiv \frac{d}{dz}. \qquad \ldots(2.9)$$

Putting $a = 1$ in (2.8), we get [Mittal 1977, p. 24, eqn. (2.6)]

$$\exp(t\, T_b)\{f(z)\} = (1 - zt)^{-b} f(z(1 - zt)^{-1}) \qquad \ldots(2.10)$$

or equivalently [Mittal 1977, eqn. (3.1)],

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_b \{f(z)\} = (1 - zt)^{-b} f(z(1 - zt)^{-1}). \qquad \ldots(2.11)$$

For $r = 1 = c_1$, $a_1 = 0 = b = m_1$, our formula (2.7) yields the following result due to Stephens (1937):

$$e^{tzD}\{f(z)\} = \lim_{a\to 0} f(z(1 - z^a t)^{-1/a})$$

$$= f(ze^t), \; D \equiv \frac{d}{dz}. \qquad \ldots(2.12)$$

Again for $r = a_1 = 1 = c_1$, $b = 0 = m_1$, our formula (2.7) reduces to another result of Stephens

$$\exp(tz^2D)\{f(z)\} = f(z(1-zt)^{-1}),\ d \equiv \frac{d}{dz}. \qquad \ldots(2.13)$$

Many other operational generating relations involving the operator ${}_aT_b$ obtained by Joshi and Prajapat (1974, 1975, 1977), and Mittal (1977) can easily be extended for the operator ${}^{a_1,\ldots,a_r}_{c_1,\ldots,c_r}U_b$. However, for the brevity of the paper, we derive here only one such relation.

We know that Erdélyi [1953, p. 101, eqn. (5)]

$$ {}_2F_1\left[\begin{matrix} a, a+1/2; \\ 1/2 \end{matrix}\ z\right] = \frac{1}{2}(1+z^{1/2})^{-2a} + \frac{1}{2}(1-z^{1/2})^{-2a}. \qquad \ldots(2.14)$$

Now, since

$$\sum_{n=0}^{\infty} \frac{t^n}{(2n)!}\ {}^{a_1,\ldots,a_r}_{c_1,\ldots,c_r}U_b^{2n}\{F(\lambda_1,\ldots,z_r)\}$$

$$= \sum_{k_1,\ldots,k_r}^{\infty} (\lambda)_M A(k_1,\ldots,k_r)\, z_1^{k_1}\ldots z_r^{k_r}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left[t\left\{\left(\prod_i z_i^{a_i}\right)\left(\sum_j c_j a_j\right)\right\}^2\right]\left(\frac{b+\sum_p c_p k_p}{\sum_l c_l a_l}\right)_{2n} \text{(by (2.2))}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_M A(k_1,\ldots,k_r)\, z_1^{k_1}\ldots z_r^{k_r}$$

$$\times\ {}_2F_1\left[\begin{matrix} \dfrac{b+\sum_p c_p k_p}{2\sum_l c_l a_l}, \dfrac{b+\sum_p c_p k_p}{2\sum_l c_l a_l}+\dfrac{1}{2}; \\ \dfrac{1}{2}; \end{matrix}\ t\left\{\left(\prod_i z_i^{a_i}\right)\left(\sum_j c_j a_j\right)\right\}^2\right]$$

in view of (2.14), we get the result

$$\sum_{n=0}^{\infty} \frac{t^n}{(2n)!}\ {}^{a_1,\ldots,a_r}_{c_1,\ldots,c_r}U_b^{2n}\{F(\lambda; z_1,\ldots,z_r)\}$$

$$= \frac{1}{2}u^b F(\lambda; z_1 u^{c_1},\ldots,z_r u^{c_r}) + \frac{1}{2}v^b F(\lambda; z_1 v^{c_1},\ldots,z_r v^{c_r}), \qquad \ldots(2.15)$$

where

$$u = \left[1 + \left\{ \left(\prod_i z_i^{a_i} \right) \left(\sum_j c_j a_j \right) \right\} t^{1/2} \right]^{-1/\Sigma_l c_l a_l}$$

and

$$v = \left[1 - \left\{ \left(\prod_i z_i^{a_i} \right) \left(\sum_j c_j a_j \right) \right\} t^{1/2} \right]^{-1/\Sigma_l c_l a_l} .$$

For $r = 1 = c_1 = a_1$ and $m_1 = 0$, our formula (2.15) reduces to [Mittal 1977, p. 26, eqn. (2.15)]

$$\sum_{n=0}^{\infty} \frac{t^n}{(2n)!} \; T_b^{2n} \{f(z)\} = \frac{1}{2} \; (1 + zt^{1/2})^{-b} f\left(\frac{z}{1 + zt^{1/2}} \right) + \frac{1}{2} \; (1 - zt^{1/2})^{-b} f\left(\frac{z}{1 - zt^{1/2}} \right). \qquad \ldots(2.16)$$

3. A Characterization For $F(\lambda\, ; z_1, \ldots, z_r)$

From (2.2), we easily obtain the formula

$$\underset{m_1, \ldots, m_r}{\overset{(rm_1)^{-1}, \ldots, (rm_r)^{-1}}{}} U_\lambda^n \; F(\lambda\, ; z_1, \ldots, z_r) = \left\{ \prod_i z_i^{(rm_i)-1} \right\}^n (\lambda)_n \, F(\lambda + n\, ; z_1, \ldots, z_r) , \qquad \ldots(3.1)$$

where $F(\lambda\, ; z_1, \ldots, z_r)$ is defined by (1.1).
Now, in view of (3.1), we can write

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; F(\lambda + n\, ; z_1, \ldots, z_r)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{\left[t \left\{ \prod_i z_i^{(rm_i)-1} \right\}^{-1} \right]^n}{n!} \times \underset{m_1, \ldots, m_r}{\overset{(rm_1)-1, \ldots, (rm_r)^{-1}}{}} U_\lambda^n \; F(\lambda\, ; z_1, \ldots, z_r).$$

Summing the right-hand side with the help of (2.3), we obtain the generating relation

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; F(\lambda + n\, ; z_1, \ldots, z_r)\, t^n$$

$$= (1 - t)^{-\lambda} F \left(\lambda\, ; \frac{z_1}{(1 - t)^{m_1}}, \ldots, \frac{z_r}{(1 - t)^{m_r}} \right) \qquad \ldots(3.2)$$

which is due to Karlsson (1975) {see also Srivastava (1977) for references to much more general results than those derived by Karlsson (1975)}. Obviously, (3.1) may also be obtained from (3.2) by comparing the later with (2.3). We have, thus, established the following characterization for $F(\lambda\,;z_1,\ldots,z_r)$:

Theorem 3.1—A necessary and sufficient condition that $F(\lambda\,;z_1,\ldots,z_r)$ be defined by the generating relation (3.2) is that it be given by (3.1).

To exhibit the importance of Theorem 3.1, we consider $F_A^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$, which are Lauricella's hypergeometric functions of r variables, of first, third and fourth kinds respectively (cf. Lauricella 1893, p. 113).

Putting $m_i = 1$ for all $i \in \{1, 2, \ldots, r\}$ and $A(k_1, \ldots, k_r)$

$= \prod_{i=1}^{r} \frac{(\alpha_i)_{k_i}}{(\beta_i)_{k_i} k_i!}$ in (1.1), we get from (3.1) the formula

$$U_{\lambda}^{n}{}^{1/r,\ldots,1/r}_{1,\ldots,1} F_A^{(r)}(\lambda\,;\alpha_1,\ldots,\alpha_r\,;\beta_1,\ldots,\beta_r\,;z_1,\ldots,z_r)$$

$$=\left\{\prod_i z_i^{1/r}\right\}^n (\lambda)_n F_A^{(r)}(\lambda+n\,;\alpha_1,\ldots,\alpha_r\,;\beta_1,\ldots,\beta_r\,;z_1,\ldots,z_r)\,. \qquad \ldots(3.3)$$

In view of (3.3), we have, by using Theorem 3.1, the generating relation

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_A^{(r)}(\lambda+n\,;\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r\,;z_1,\ldots,z_r)\,t^n$$

$$=(1-t)^{-\lambda} F_A^{(r)}\left(\lambda\,;\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r\,;\frac{z_1}{1-t},\ldots,\frac{z_r}{1-t}\right) \qquad \ldots(3.4)$$

provided $\max.\{\,|z_1/(1-t)|+\ldots+|z_r/(1-t)|\,,\,|t|\,\}<1$. Similarly,

when $A(k_1,\ldots,k_r = \frac{(\alpha)_{k_1+\ldots+k_r}}{\prod_{i=1}^{r}(\beta_i)_{k_i}(k_i)!}$, and

$A(k_1,\ldots,k_r) = \frac{1}{(\beta)_{k_1+\ldots+k_r}} \prod_{i=1}^{r}\frac{(\alpha_i)_{k_i}}{(k_i)!}$, and of course $m_i = 1$ for all $i \in \{1, 2, \ldots, r\}$, we get the generating relations

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_C^{(r)}(\lambda+n,\alpha\,;\beta_1,\ldots,\beta_r\,;z_1,\ldots,z_r)$$

$$=(1-t)^{-\lambda} F_C^{(r)}\left(\lambda,\alpha\,;\beta_1,\ldots,\beta_r\,;\frac{z_1}{1-t},\ldots,\frac{z_r}{1-t}\right), \qquad \ldots(3.5)$$

where $\sqrt{|z_1/(1-t)|}+\ldots+\sqrt{|z_r/(1-t)|}<1$ and $|t|<1$, and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_D^{(r)}\left(\lambda+n\,;\alpha_1,\ldots,\alpha_r\,;\beta;z_1,\ldots,z_r\right)t^n$$

$$=(1-t)^{-\lambda} F_D^{(r)}\left(\lambda\,;\alpha_1,\ldots,\alpha_r\,;\beta;\frac{z_1}{1-t},\ldots,\frac{z_r}{1-t}\right) \quad \ldots(3.6)$$

where $|t|$, $|z_i/(1-t)|<1$, $i=1,\ldots,r$; respectively.

Note that (3.4), (3.5) and (3.6) were also obtained by Srivastava (1972, pp. 78–79) by direct summation. In fact, Srivastava [1972 p. 79, eqn. (2.8)] gave a much more general result than (3.4), (3.5) and (3.6) above, which involves the generalized Lauricella function defined by Srivastava and Daoust (1969, p. 454).

In conclusion, it is worthwhile to remark that the formula (2.11) is the base of the works by Joshi and Prajapat, Mittal, and others. Many of the results obtained by them in the various papers cited are the consequences of this very formula. Thus the present work, which generalizes this formula, is appropriately termed as the multivariable extension of the works referred to above.

Acknowledgement

The authors are indebted to Prof. H. M. Srivastava for his suggestions which led to a better presentation of the paper.

References

Erdélyi, A *et al.* (1953). Higher Transcendetal Functions, Vol. 1. McGraw Hill Book Co., Inc. New York.

Joshi, C. M., and Prajapat, M. L. (1974). On some properties of a class of polynomials unifying the generalized Hermite, Laguerre and Bessel polynomials II. *Proc. Nat. Acad. Sci. India Sect. A*, **44**, 311–18.

——— (1975). The operator $T_{k,q}$ and a generalization of certain classical polynomials. *Kyungpook Math. J.*, **15**, 191–99.

——— (1977). On some properties of a class of polynomials unifying the generalized Hermite, Laguerre and Bessel polynomials. *Math. Student*, 45, (2), 74–86.

Karlsson, P. W. (1975). Some generating functions of several variables. *ZAMM*, **55**, 49.

Lauricella, G. (1893). Sulle funzioni ipergeometriche a più variabili. *Rend. Circ. Mat. Palermo*, 7, 111–58.

Mittal, H. B. (1972). Polynomials defined by generating relations. *Trans. Am. Math. Soc.*, **168**, 73–84.

——— (1977). Bilinear and bilateral generating relations. *Am. J. Math.*, 99, 23–55.

Pólya, G., and Szegö, G. (1925). Aufgaben and Lehrsatze aus der Analysis, I. Springer, Berlin.

Srivastava, H. M. (1972). Certain formulas involving Appell Functions. *Comment. Math. Univ. St. Paul.*, **21**, fasc. 1, 73–99.

——————(1977). Certain generating functions of several variables. *ZAMM*, **57**, 339–40.

Srivastava, H. M., and Daoust, M. C. (1969). Certain generalized Neumann expansions associated with the Kampé de Fériet function. *Nederl. Akad. Wetensch. Proc. Ser. A* 72 = *Indag. Math.* **31**, 449–57.

Stephens, E. (1937). The Elementary Theory of Operational Mathematics. McGraw Hill Book Co. Inc., New York, London.