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GENERAL FORMULAS FOR A TERMINATING GENERALISED KAMPÉ DE FÉRIET SERIES

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Two general formulas for a terminating generalised Kampé de Fériet function with unit arguments have been obtained. A number of sums have also been deduced. They include, as special cases, a sum of Carlitz (1967) and the author (1980), and a transformation formula of Singal (1971).

1. INTRODUCTION

Let $F^{1:p;q}_{1:r;s}$ denote a terminating generalised Kampé de Fériet function with unit arguments (Burchnall and Chaundy 1941) in the modified notation defined by

$$F^{1:p;q}_{1:r;s}\left[\begin{matrix}-m : \{a_p\} ; \{b_q\} \\ f : \{c_r\} ; \{d_s\}\end{matrix}\right] = \sum_{i+j\leqslant m} \frac{(-m)_{i+j} \prod_{k=1}^{p} (a_k)_i \prod_{k=1}^{q} (b_k)_j}{(f)_{i+j} \prod_{k=1}^{r} (c_k)_i \prod_{k=1}^{s} (d_k)_j \; i\,!\; j\,!} \qquad \ldots(1.1)$$

where $(\lambda)_n$ is given by

$$(\lambda)_n = \begin{cases}\dfrac{\Gamma(\lambda+n)}{\Gamma_1(\lambda)} = \lambda(\lambda+1)\ldots(\lambda+n-1), & n \neq 0 \text{ when } n = 0.\end{cases}$$

Also let ${}_pF_q$ denote the usual generalised hypergeometric function of unit argument defined by

$${}_pF_q\left[\begin{matrix}\{a_p\} ; \\ \{b_q\} ;\end{matrix}\right] = \sum_{i=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_i}{\prod_{k=1}^{q} (b_k)_i \; i\,!}$$

where $\sum_1^q b_k - \sum_1^p a_k > 0$. We assume that the parameters may be real or complex.

Sums of (1.1) for different values of $p = q$ and $r = s$ have been obtained by Carlitz (1967), Saran (1980), Singal (1971), and several others. In this paper we obtain two general formulas for the terminating generalised Kampé de Fériet function for $p = 3$, $r = 2$ and for general q and s. A number of particular cases have also been obtained which are new and include a sum of Carlitz (1967) and the author Saran (1980). Singal's transformation has also been deduced.

2. General Formulas

The general formulas to be proved are

$$F^{1:3;q+2}_{1:2;s+1}\left[\begin{matrix}-m\ ;\ b,\ 1+\frac{1}{2}b,\ \frac{1}{2}(b+f-1)+m;\ 1+f+m,\ \frac{3}{2}+\frac{1}{2}(f+b),\ \{a_q\}\\ 1+f+b+m;\quad \frac{1}{2}b,\ \frac{1}{2}(b-f+3)-m;\quad -\frac{1}{2}+\frac{1}{2}(f-b),\ \{c_s\};\end{matrix}\right]$$

$$=\frac{(f)_{2m}\,(1+f+b)_m\,(\frac{3}{2}+\frac{1}{2}f+\frac{1}{2}b)_m}{(1+f)_m\,(1+f+b)_{2m}\,(-\frac{1}{2}+\frac{1}{2}f-\frac{1}{2}b)_m}\ {}_{q+2}F_{s+1}\left[\begin{matrix}-m,\ 1+\frac{1}{2}f,\ \{a_q\};\\ \frac{1}{2}f,\ \{c_s\};\end{matrix}\right] \quad \ldots(2.1)$$

and

$$F^{1:3;q+2}_{1:3;s+1}\left[\begin{matrix}-m\ :\ b,\ 1+\frac{1}{2}b,\ \frac{1}{2}(b+f)+m;\ 1+f+m,\ 1+\frac{1}{2}(f+b),\ \{a_q\}\\ 1+f+b+m:\quad \frac{1}{2}b,\ \frac{1}{2}(2+b-f)-m;\quad \frac{1}{2}(f-b),\ \{c_s\}\end{matrix}\right]$$

$$=\frac{(1+f)_{2m}\,(1+\frac{1}{2}f+\frac{1}{2}b)_m\,(1+f+b)_m}{(1+f)_m\,[\frac{1}{2}(f-b)]_m\,(1+f+b)_{2m}}\ {}_{q+1}F_s\left[\begin{matrix}-m,\ \{a_q\};\\ \{c_s\}\end{matrix}\right] \quad \ldots(2.2)$$

where m is a non-negative integer.

The proofs of these results are simple. The given function is first expressed in terms of a series involving hypergeometric series and the known sums of nearly-poised hypergeometric series of the second kind is applied. The order of summation is then inverted to obtain the formula in the final form. We have two sums for the nearly-poised hypergeometric series due to Bailey [1964, p. 30, results (1.3) and (1.4)]

$${}_4F_3\left[\begin{matrix}b,\ 1+\frac{1}{2}b,\quad d,\quad -m;\\ \frac{1}{2}b,\ 1+b-d,\ 1+2d-m\end{matrix}\right]=\frac{(b-2d)_m\,(-d)_m}{(1+b-d)_m\,(-2d)_m} \quad \ldots(A)$$

and

$${}_4F_3\left[\begin{matrix}b,\ 1+\frac{1}{2}b,\quad d,\quad -m;\\ \frac{1}{2}b,\ 1+b-d,\ 2+2d-m\end{matrix}\right]=\frac{(b-2d-1)_m\,(-d-1)_m\,(\frac{1}{2}+\frac{1}{2}b-d)_m}{(1+b-d)_m\,(-2d-1)_m\,(\frac{1}{2}b-d-\frac{1}{2})_m}. \quad \ldots(B)$$

To prove (2.2), say, we have

$$F^{1:3;q+2}_{1:2;s+1}=\sum_{i=0}^{m}\frac{(-m)_i\,(1+f+m)_i\,[1+\frac{1}{2}(f+b)]_i\,\{a_q\}_i}{(1+f+b+m)_i\,[\frac{1}{2}(f-b)]_i\,\{c_s\}_i\,i\,!}$$

$$\times\ {}_4F_3\left[\begin{matrix}b,\ 1+\frac{1}{2}b,\ \frac{1}{2}(b+f)+m;\quad -m+i\,;\\ \frac{1}{2}b,\ \frac{1}{2}(2+b-f)-m;\ 1+b+f+m+i\end{matrix}\right].$$

The inner ${}_4F_3$ on using (A) equals

$$=\frac{(-f-2m)_{m-i}\,(-\frac{1}{2}f-\frac{1}{2}b-m)_{m-i}}{[\frac{1}{2}(2+b-f)-m]_{m-i}(-b-f-2m)_{m-i}}$$

$$=\frac{(1+f)_{2m}\,(1+\frac{1}{2}f+\frac{1}{2}b)_m\,(1+f+b)_m\,[\frac{1}{2}(f-b)]_i\,(1+b+f+m)_i}{(1+f)_m\,[\frac{1}{2}(f-b)]_m\,(1+f+b)_{2m}\,(1+f+m)_i\,(1+\frac{1}{2}b+\frac{1}{2}f)_i}.$$

Applying this value and then changing the order of summation (which is valid as the series is terminating) we immediately prove the result. Similarly (2.1) can be proved by using (B).

3. Particular Cases

A large number of sums of generalised Kampé de Fériet function can be obtained by using known sums for various values of q and s. For example, from (2.1) we have

$$F^{1:3;4}_{1:2;3}\left[\begin{matrix} -m: b,\ 1+\frac{1}{2}b,\ \frac{1}{2}(b+f-1)+m; & f, & b, & d, & e \\ 1+f+b+m: \quad \frac{1}{2}b,\ \frac{1}{2}(b-f+1)-m; & 1+f-b, & 1+f-d, & 1+f-e \end{matrix}\right]$$

$$= \frac{(f)_{2m}\,(1+f+b)_m\,(\frac{3}{2}+\frac{1}{2}f+\frac{1}{2}b)_m}{(1+f)_m\,(1+f+b)_{2m}\,(-\frac{1}{2}+\frac{1}{2}f-\frac{1}{2}b)_m}$$

$$\times\ {}_7F_6\left[\begin{matrix} f,\ 1+\frac{1}{2}f, & b, & d, & e, & -\frac{1}{2}+\frac{1}{2}(f-b) & -m; \\ \frac{1}{2}f, & 1+f-b, & 1+f-d, & 1+f-e, & \frac{3}{2}+\frac{1}{2}(f+b), & 1+f+m \end{matrix}\right].$$

Using Dougall's formula [Bailey (1964), result 4.3.5), p. 26] we get

$$F^{1:3;4}_{1:2;3}\left[\begin{matrix} -m: b,\ 1+\frac{1}{2}b,\ \frac{1}{2}(b+f-1)+m; & f, & b, & d, & e \\ 1+f+b+m: \quad \frac{1}{2}b,\ \frac{1}{2}(b-f+1)-m; & 1+f-b, & 1+f-d, & 1+f-e \end{matrix}\right]$$

$$= \frac{(f)_{2m}\,(1+f+b)_m\,(\frac{3}{2}+\frac{1}{2}f+\frac{1}{2}b)_m\,(1+f-b-e)_m\,(1+f-b-d)_m\,(1+f-e-d)_m}{(1+f+b)_{2m}\,(-\frac{1}{2}+\frac{1}{2}f-\frac{1}{2}b)_m(1+f-b)_m(1+f-e)_m\,(1+f-d)_m(1+f-b-e-d)_m} \qquad \ldots(3.1)$$

provided $\frac{1}{2}(3+b+3f) = b+d+e-m$.

Also from (2.1) we have if $a_q = \frac{1}{2}f,\ c_s = 1+\frac{1}{2}f$

$$F^{1:3;3}_{1:2;2}\left[\begin{matrix} -m: b,\ 1+\frac{1}{2}b,\ \frac{1}{2}(b+f-1)+m; & 1+f+m,\ \frac{3}{2}+\frac{1}{2}(f+b), & \frac{1}{2}f \\ 1+f+b+m: \quad \frac{1}{2}b,\ \frac{1}{2}(b-f+3)-m; & -\frac{1}{2}+\frac{1}{2}(f-b), & 1+\frac{1}{2}f \end{matrix}\right]$$

$$= \frac{(f)_{2m}\,(1+f+b)_m\,(\frac{3}{2}+\frac{1}{2}f+\frac{1}{2}b)_m}{(1+f)_m\,(1+f+b)_{2m}\,(-\frac{1}{2}+\frac{1}{2}f-\frac{1}{2}b)_m}. \qquad \ldots(3.2)$$

The result of the author (Saran 1980) can be proved from (2.2) if we take $a_q = \frac{1}{2}(f-b),\ d,\ f,\ 1+\frac{1}{2}f$ and $c_s = 1+f+m,\ \frac{1}{2}f,\ 1+\frac{1}{2}(f+b),\ 1+f-d$ and sum the inner function by Bailey's result (1964). If, however, we sum the R. H. S. of (2.1) by Bailey's formula (1964) we get a new result

$$F^{1:3;2}_{1:2;1}\left[\begin{matrix} -m: b,\ 1+\frac{1}{2}b,\ \frac{1}{2}(b+f-1)+m; & f, & d \\ 1+f+b+m: \quad \frac{1}{2}b,\ \frac{1}{2}(b-f+3)-m; & & 1+f-d \end{matrix}\right]$$

$$= \frac{(f)_{2m}\,(1+f+b)_m\,(\frac{3}{2}+\frac{1}{2}f+\frac{1}{2}b-d)_m}{(1+f+b)_{2m}\,(1+f-d)_m\,(-\frac{1}{2}+\frac{1}{2}f+\frac{1}{2}b)_m}. \qquad \ldots(3.3)$$

Carlitz's result can be deduced from (3.1) if we put down the value of b from the given condition and let e tend to infinity. Singal's result [(1971), result (1.1), p. 610] can be deduced from (2.1) by taking $-m = d$, $1 + f + b + m = e$, b for a $1+\frac{1}{2}b$ for b, $\frac{1}{2}(b+f-1)+m$ for c, $\frac{3}{2}+\frac{1}{2}(f+b)$ for b' and let $\{a_q\}$ stand for f, c' and $\{b_s\}$ for $1+f+m$, $1+f-c'$ and the using Whipple's transformation [Bailey (1964), § 4.5 (I), p. 30].

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