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ON THE DEGREE OF APPROXIMATION OF A PERIODIC FUNCTION f BY ALMOST RIESZ MEANS OF ITS CONJUGATE SERIES

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In the present paper the author has obtained the degree of approximation of certain functions belonging to the class Lip α by almost Riesz means.

§1. Let f be a 2π periodic function integrable $L^p (p>1)$ and let

$$f \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{...(1.1)}$$

be its Fourier series.

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx). \qquad \text{...(1.2)}$$

A function $f \in$ Lip α if

$$f(x+h)-f(x)=O(|h|^{\alpha}) \text{ for } 0<\alpha\leqslant 1. \qquad \text{...(1.3)}$$

Lorentz (1948) has defined:

Definition L_1—A sequence $\{s_n\}$ is said to be almost convergent to a limit S, if

$$\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{k=p}^{n+p} S_k = S \qquad \text{...(1.4)}$$

uniformly with respect to p.

An almost convergence is a generalization of ordinary convergence.

Sharma and Qureshi (1980) defined :

Definition L_2—A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{s_n\}$ is said to be almost Riesz summable to S, provided

$$T_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_{k,p} \to S \text{ as } n \to \infty \qquad \text{...(1.5)}$$

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uniformly with respect to p, where

$$S_{k,p} = \frac{1}{k+1} \sum_{\mu=1}^{k+p} S_\mu$$

and $\{p_n\}$ be a sequence of non-negative constants, such that $p_0 > 0$ and

$P_n = p_0 + p_1 + \ldots + p_n.$

The Riesz means are regular if and only if $P_n \to \infty$ with n [see Theorem 1.4.4 of Petersen (1966)].

§ 2. Sharma and Qureshi (1980) proved the following theorem.

Theorem A—The degree of approximation of a periodic function f with period 2π and belonging to the class of Lip α by almost Riesz means of its Fourier series is given by

$$\max_{0 \leqslant x \leqslant 2\pi} |f(x) - T_{n,p}(x)| = \begin{cases} O\left\{\left(\frac{p_n}{P_n}\right)^\alpha\right\}; & 0 < \alpha < 1 \\ O\left\{\frac{p_n}{P_n} \log \frac{P_n}{p_n}\right\}; & \alpha = 1 \end{cases}$$

where Riesz means are regular and $0 < p_n \uparrow$ with $n \geqslant n_0$

Object of this paper is to prove the following theorem.

Theorem B—The degree of approximation of a periodic function $\bar{f}(x)$, conjugate to a 2π periodic function f and belonging to the class of Lip α by almost Riesz means of its conjugate series, is given by

$$\max_{0 \leqslant x \leqslant 2\pi} |f(x) - \overline{T_{n,p}}(x)| = \begin{cases} O\left\{\left(\frac{p_n}{P_n}\right)^\alpha\right\}; & 0 < \alpha < 1 \\ O\left\{\frac{p_n}{P_n} \log \frac{P_n}{p_n}\right\}; & \alpha = 1 \end{cases}$$

where $\overline{T_{n,p}}(x)$ are the almost Riesz mean of the series (1.2) and Riesz means are regular such that $0 < p_n \uparrow$ with $n \geqslant n_0$.

Proof of the Theorem: Let $\overline{S_k}$ be the k-th partial sum of the conjugate series (1.2), it is easy to show that

$$\overline{S_k}(x) - \bar{f}(x) = -\frac{1}{\pi} \int_0^\pi \psi(t) \frac{\cos(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}\, dt$$

where

$$\psi(t) = f(x+t) - f(x-t)$$

and

$$\overline{S_{k,p}}(x) - \bar{f}(x) = \frac{1}{k+1} \sum_{\mu=p}^{k+p} \left\{\overline{S_\mu}(x) - \bar{f}(x)\right\}$$

(*equation continued on p.* 1138)

$$= -\frac{1}{\pi(k+1)}\int_0^\pi \psi(t)\sum_{\mu=p}^{k+p}\frac{\cos(\mu+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{2\pi(k+1)}\int_0^\pi \psi(t)\frac{\sin pt-\sin(k+p+1)t}{2\sin^2\frac{1}{2}t}\,dt.$$

We have

$$\overline{t_{n,p}}(t)-\overline{f}(t)=\frac{1}{P_n}\sum_{k=0}^{n}p_k\left\{\overline{S_{k,p}}(t)-\overline{f}(t)\right\}$$

$$= \frac{1}{2\pi P_n}\int_0^\pi \psi(t)\sum_{k=0}^{n}\frac{p_k}{k+1}\frac{[\sin pt-\sin(k+p+1)t]}{2\sin^2\frac{1}{2}t}\,dt,$$

Therefore

$$\left|\overline{t_{n,p}}(t)-\overline{f}(t)\right| \leqslant \frac{1}{2\pi P_n}\int_0^\pi |\psi(t)|\left|\sum_{k=0}^{n}\frac{p_k}{k+1}\frac{\cos(k+2p+1)\frac{1}{2}t\,\sin(k+1)\frac{1}{2}t}{\sin^2\frac{1}{2}t}\right|dt$$

$$= \frac{1}{2\pi P_n}\left[\int_0^{p_n/P_n}+\int_{p_n/P_n}^{\pi}\right]|\psi(t)|\left|\sum_{k=0}^{n}\frac{p_k}{k+1}\right.$$

$$\left.\times\frac{\cos(k+2p+1)\frac{1}{2}t\,\sin(k+1)\frac{1}{2}t}{\sin^2\frac{1}{2}t}\right|dt$$

$$= I_1+I_2, \text{ say.}$$

Now

$$I_1 = O\left[\frac{1}{P_n}\int_0^{p_n/P_n} t^\alpha\sum_{k=0}^{n}\frac{p_k}{k+1}\frac{|\cos(k+2p+1)\frac{1}{2}t\,\sin(k+1)\frac{1}{2}t|}{\sin^2\frac{1}{2}t}\,dt\right]$$

$$= O\left[\frac{1}{P_n}\int_0^{p_n/P_n} t^\alpha\sum_{k=0}^{n}\frac{p_k}{k+1}\,\frac{k+1}{t}\,dt\right]$$

$$= O\left[\int_0^{p_n/P_n} t^{\alpha-1}\,dt\right]$$

$$= O\left[\left(\frac{p_n}{P_n}\right)^\alpha\right].$$

Also

$$I_2 = O\left[\frac{1}{P_n} \int_{p_n/P_n}^{\pi} t^{\alpha} \left| \sum_{k=0}^{n} \frac{p_k}{k+1} \frac{\cos(k+2p+1)\frac{1}{2}t \sin(k+1)\frac{1}{2}t}{\sin^2 \frac{1}{2}t} \right| dt \right]$$

$$= O\left[\frac{1}{P_n} \int_{p_n/P_n}^{\pi} t^{\alpha} \sum_{k=0}^{n} \frac{1}{(k+1)} \frac{(k+1) \mid \sin\frac{1}{2}t \mid \; \mid p_k \cos(k+2p+1)\frac{1}{2}t \mid}{\sin^2 \frac{1}{2}t} dt \right]$$

$$= O\left[\frac{1}{P_n} \int_{p_n/P_n}^{\pi} t^{\alpha-1} \sum_{k=0}^{n} \left| p_k \cos(k+2p+1)\frac{1}{2}t \right| dt \right]$$

$$= O\left[\frac{p_n}{P_n} \int_{p_n/P_n}^{\pi} t^{\alpha-2} \, dt \right]$$

since $\{p_n\}$ is monotonic increasing, we have

$$\sum_{k=0}^{n} p_k \cos(k+2p+1)\tfrac{1}{2}t \leqslant p_n \sum_{k=0}^{n} \cos(k+2p+1)\tfrac{1}{2}t$$

$$= O\left(\frac{p_n}{t} \right).$$

Therefore, in the case $0<\alpha<1$, we have

$$I_2 = O\left[\frac{p_n}{P_n} \left(\frac{p_n}{P_n} \right)^{\alpha-1} \right]$$

$$= O\left[\left(\frac{p_n}{P_n} \right)^{\alpha} \right]$$

and in the case $\alpha = 1$, we have

$$I_2 = O\left[\frac{p_n}{P_n} \log \frac{P_n}{p_n} \right].$$

This completes the proof of the theorem.

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