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BITOPOLOGICAL SPACES AND ASSOCIATED Q-PROXIMITY

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In this paper implications have been studied between various separation axioms in a bitopological space. Investigations have been made about nature of q-proximity compatible with a given bitopological space, and it has been proved among other results that there is a unique q-proximity compatible with a pairwise compact and pairwise regular bitopological space.

§1. Kelly (1963) has studied consequences of introducing a q-metric (quasi-metric) i.e., a metric minus axiom of symmetry in a non-empty set X. Given a q-metric p on X there is another such p_* on X called the conjugate of p given by $p_*(x, y) = p(y, x)$ for all $x, y \in X$. There results a bitopological space (X, p, p_*). Lane (1967) has developed in good length theory of q-uniform structure and proved that a bitopological space is q-uniformizable iff it is pairwise completely regular. The first author with Banerjee have proved in Jas and Banerjee (1981) that a bitopological space is q-proximizable iff it is pairwise completely regular. In this paper we have studied interaction of various separation axioms in a bitopological space, and also studied consequences of q-proximity in a bitopological space. It has been shown that a pairwise normal bitopological space with both topologies T_1 is q-proximizable. While searching for a unique q-proximity compatible with a given bitopological space we have found that a pairwise compact and pairwise regular bitopological space admits of a unique compatible q-proximity, the formulation of which has also been laid down.

§2. It is necessary to recall the following definition as in Kelly (1963).

Definition 1.1—In a bitopological space (X, τ, V), τ is said to be regular with respect to V if for each x in X there is a τ-neighbourhood base of V-closed sets or equivalently if for x in X and each τ-closed set F with $x \notin F$ there are τ-open set G and V-open set H such that $x \in G$ and $F \subset H$ and that $G \cap H = \emptyset$. (X, τ, V) is said to be pairwise regular if τ is regular w.r.t. V and V is regular w.r.t. τ. Definitions of pairwise normal and pairwise Hausdorff bitopological spaces are now clear.

Definition 1.2—In a bitopological space (X, τ, V), τ is said to be completely regular w.r.t. V if for each point x and each τ-open neighbourhood U of x, there is a τ-lower semi-continuous (l.s.c.) and V-upper semi-continuous (u.s.c.) function $f: X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) = 0$. (X, τ, V) is said to be pairwise completely regular if τ is completely regular w.r.t. V and V is completely regular w.r.t τ.

Definition 1.3 [Fletcher *et al.* (1969)]—A cover $\mathscr{U}$ of a bitopological space (X, τ, V) is pairwise open if $\mathscr{U} \subset \tau \cup V$ with $\mathscr{U} \cap \tau$ containing a non-empty set, and with $\mathscr{U} \cap V$ containing a non-empty set.

Definition 1.4 (see Fletcher *et al.* 1969)—A bitopological space (X, τ, V) is called pairwise compact if every pairwise open cover of X has a finite subcover. Following definition 1.4 Fletcher *et al.* (1969) have proved the following theorems.

Theorem 1.1 [Theorem 12 of Fletcher *et al.* (1969)]

If (X, τ, V) is pairwise Hausdorff and pairwise compact, then (X, τ, V) is pairwise regular.

Theorem 1.2 [Theorem 13 of Fletcher *et al.* (1969)]—If (X, τ, V) is pairwise compact and either τ is regular w.r.t. V or V is regular w.r.t. τ, then (X, τ, V) is pairwise normal.

Corollary 1.1—Every pairwise compact and pairwise regular bitopological space is pairwise normal.

Weakening pairwise compactness to pairwise Lindelöff (i.e., each pairwise open cover admits of a countable sub-cover) property we prove the following theorem :

Theorem 1.3—If (X, τ, V) is pairwise regular and pairwise Lindelöff, then (X, τ, V) is pairwise normal.

PROOF : Let A be a τ-closed set and B be a V-closed set with $A \cap B = \emptyset$. Let $x \in A$, by pairwise regularity there is a V-open set $N_x(A)$ satisfying $x \in N_x(A) \subset \tau\text{-cl}(N_x(A)) \subset X \setminus B$. Then $\{N_x(A)\}_{x \in A} \cup (X \setminus A)$ is a pairwise open cover of X. Similarly for each $y \in B$, there is a τ-open set $N_y(B)$ satisfying $y \in N_y(B) \subset V\text{ cl}(N_y(B)) \subset X/A$ and $\{N_y(B)\}_{y \in B} \cup (X/B)$ is a pairwise open cover of X. Rest of the proof follows the pattern of the proof of classical Lemma (Kelley 1955, p.113) and proceeding as therein we find the desired open sets to separate A and B.

Theorem 1.4—Product of pairwise completely regular bitopological spaces is pairwise completely regular.

PROOF : Let $(X_\alpha, \tau_\alpha, V_\alpha)_{\alpha \in \Delta}$ be a family of bitopological spaces. Let $X = X\{X_\alpha : \alpha \in \Delta\}$, and $\tau = X\{\tau_\alpha : \alpha \in \Delta\}$ and $V = X\{V_\alpha : \alpha \in \Delta\}$. For $x \in X$ and $U \in \tau$ with $x \in U$, let us call a τ-l.s.c. V-u.s.c, function $f : X \rightarrow [0, 1]$ meant for (x, U) whenever $f(x) = 1$ and $f(X \setminus U) = 0$. If $f_1, f_2, \ldots, f_n$ are functions meant for (x, U_1), $(x, U_2), \ldots (x, U_n)$ and if $g(x) = \sup\{f_i(x) : i = 1, 2, \ldots, n\}$, then g is a function meant for $(x, \cap\{U_i : i = 1, 2, \ldots, n\})$. Consequently in (X, τ, V), τ is completely regular w.r.t. V if for each $x \in X$ and each τ-open neighbourhood U of x belonging to some sub-base for the topology τ, there is a function f meant for (x, U). Let $x \in U$ and $U_\alpha \in \tau_\alpha$ be a neighbourhood of x_α in X_α, and let f be the function meant for (x_α, U_α); then $f_0 P_\alpha$ where P_α is the projection mapping onto X_α becomes a function meant for $\left(x, P_\alpha^{-1}(U_\alpha)\right)$. Since $\left\{P_\alpha^{-1}(U_\alpha)\right\}$ form a sub-base for τ, we have shown that τ is completely regular w.r.t. V. Similarly we show that V is completely regular w.r.t. τ. The proof is now complete.

Theorem 1.5—Every pairwise compact and pairwise regular bitopological space is pairwise completely regular.

Proof : Suppose a bitopological space (X, τ, V) is pairwise compact and pairwise regular. Then by Theorem 1.2, (X, τ, V) is pairwise normal. Let $G \in \tau$ and $x \in G$. By pairwise regularly we have a V-closed τ-open neighbourhood F of x such that $x \in F \subset G$. So $F \cap (X \setminus G) = \emptyset$. Further $X \setminus G$ is τ-closed. By pairwise normality, we apply Theorem 2.7 of Kelly (1963) to find a τ-u.s.c and V-l.s.c. function $f: X \to [0, 1]$ such that $f(X \setminus G) = 1$ and $f(F) = 0$. This gives $f(x) = 0$ and $f(X \setminus G) = 1$; and hence τ is completely regular w.r.t. V. Similarly we show that V is completely regular w.r t. τ. The proof is now complete.

Theorem 1.6—Let (X, τ, V) has a subset Y which is (i) both τ- and V-closed and (ii) both τ- and V-discrete. If Y has a subset D dense w.r.t. upper bound topology of τ and V, and $|Y| \geqslant 2^{|D|}$, then (X, τ, V) is not pairwise normal.

(| | denoting the cardinality). Proof goes the same way as in Dugundji (1975) and is left out.

Example of a pairwise completely regular bitopological space which is not pairwise normal.

Example 1.1—Let $\mathscr{R}$ be the space of reals with topologies $\mathscr{U}$ and $\mathscr{V}$, where $\mathscr{U}$ has $(a, b]$ as basic open sets, and $-\mathscr{V}$ has $[a, b)$, $a < b$ as basic open sets. It is easy to see that $(\mathscr{R}, \mathscr{U}, \mathscr{V})$ is pairwise regular and pairwise Lindelöff. By Theorem 1.3 $(\mathscr{R}, \mathscr{U}, \mathscr{V})$ is pairwise normal. Since both topologies $\mathscr{U}$ and $\mathscr{V}$ are T_1, it follows that $(\mathscr{R}, \mathscr{U}, \mathscr{V})$ is pairwise completely regular. Let $X = \mathscr{R} \times \mathscr{R}$, $\tau = \mathscr{U} \times \mathscr{U}$, and $V = \mathscr{V} \times \mathscr{V}$. Then by Theorem 1.4 (X, τ, V) is pairwise completely regular. Let $Y = \{(x, -x) : x \in \mathscr{R}\}$. Then $Y \subset X$ has a subset $D = \{(x, -x) : x$ is rational$\}$ such that Y and D satisfy all the conditions of Theorem 1.6 an application of which shows that (X, τ, V) is not pairwise normal.

§3. In this section we study consequences of q-proximity in a bitopological space in context of various separation axioms. A q-proximity is a proximity minus the requirement of symmetry, the definition of which is given below.

There is a kind of proximity introduced by Pervin in Naimpally and Warrack (1970) and we now remark that definition of Pervin proximity and q-proximity as given by us in Definition 2.1 are equivalent. We recall from Jas and Banerjee (1981) that a bitopological space (X, τ, V) is pairwise completely regular if and only if there is a compatible q-proximity $\prec$ i.e , $\prec$ satisfying $\tau = \tau_{\prec}$ and $V = \tau_{\prec *}$ where $\tau_{\prec}$ and $\tau_{\prec *}$ are topologies induced by $\prec$ and its conjugate q-proximity $\prec^*$ given by $A \prec^* B$ iff $(X \setminus B) \prec (X \setminus A)$. The proof of this result owes to the fact that if $A \prec B$ then there is a $\tau_{\prec}$ -u.s.c. and $\tau_{\prec *}$-l.s.c. function $f: X \to [0,1]$ such that $f(A)=0$ and $f(X \setminus B)=1$. This result also appears in Jas and Banerjee (1981).

Definition 2.1—A relation on the power set of X denoted by $\prec$ is said to define a q-proximity on X $(\neq \emptyset)$ iff it satisfies the following axioms :

Q 1. $X \prec X$ and $\emptyset \prec \emptyset$

Q 2. $A \prec B$ implies $A \subset B$

Q 3. $A \subset B \prec C \subset D$ implies $A \prec D$

Q 4. $A \prec B_K$; $K = 1, 2\ldots, n$ implies $A \prec \bigcap_{K=1}^{n} B_K$

$$A_k \prec B,\ K = 1, 2\ldots, n \text{ implies } \bigcup_{k=1}^{n} A_K \prec B$$

and

Q 5. $A \prec B$ implies the existence of a subset C of X such that $A \prec C \prec B$.

We now quote the following theorem which is converse of Theorem 2.7 of Kelly (1963).

Theorem 2.1—Let A be τ-closed and B be $\mathbf{V}$-closed sets in $(X, \tau, \mathbf{V})$ such that $A \cap B = \emptyset$. If there is a τ-u.s.c. and $\mathbf{V}$-l.s.c. function $f: X \to [0, 1]$ such that $f(A) = 1$ and $f(B) = 0$ then $(X, \tau, \mathbf{V})$ is pairwise normal.

The proof is a routine work and is left out.

Suppose $\prec$ is a q-proximity to make a bitopological space $(X, \tau, \mathbf{V})$ q-proximizable, and suppose $\mathbf{V}\text{-cl}(A) \cap \tau\text{-cl}(X \setminus B) = \emptyset$ for two subsets A and B of X. It does not necessarily follow that $A \prec B$. Because otherwise by Theorem 2.1 and by what has been stated above we would have $(X, \tau, \mathbf{V})$ as pairwise normal.

Example 1.1 above confirms of a contrary situation.

Theorem 2.2—If $(X, \tau, \mathbf{V})$ is pairwise normal with both topologies T_1, then a compatitble q-proximity $\prec$ on X is given by $A \prec B$ iff $\mathbf{V}\text{-cl}(A) \cap \tau\text{-cl}(X/B) = \emptyset$.

PROOF : It is a routine exercise to show that $\prec$ defined by the prescribed rule is indeed a q-proximity on X. We next show that $\tau = \tau_\prec$ and $\mathbf{V} = \tau_{\prec *}$.

Let G be a $\tau_\prec$-open set and $x \in G$. Since $\mathbf{V}$ is T_1, $\{x\}$ is $\mathbf{V}$-closed; Also $\{x\} \prec G$. So we have $\{x\} \cap \tau\text{-cl}(X/G) = \emptyset$. This gives $x \in \tau\text{-int}(G)$, i.e. $G \subset \tau\text{-int}(G)$. So G is τ-open. Conversely let A be τ-open, and $x \in A$: since $\{x\} \cap (X/A) = \emptyset$; i.e. $\mathbf{V}\text{-cl}\{x\} \cap \tau\text{-cl}(X \sim A) = \emptyset$. We have $\{x\} \prec A$. Hence $x \in \tau_\prec\text{-int}(A)$; i.e. $A \subset \tau_\prec\text{-int}(A)$. Thus A is $\tau_\prec$-open. Using $A \prec^* B$ iff $(X \setminus B) \prec (X \setminus A)$, iff $\tau\text{-cl}(A) \cap \mathbf{V}\text{-cl}(X \setminus B) = \emptyset$, we proceed similar to above to show that $\tau_{\prec *} = \mathbf{V}$.

Theorem 2.3—If a pairwise completely regular bitopological space $(X, \tau, \mathbf{V})$ has a compatible q-proximity $\prec$ given by $A \prec B$ iff $\mathbf{V}\text{-cl}(A) \cap \tau\text{-cl}(X \setminus B) = \emptyset$, then $(X, \tau, \mathbf{V})$ is pairwise normal.

PROOF : Let C be a τ-closed set and D be a $\mathbf{V}$-closed set which $C \cap D = \emptyset$. Hence $D \prec (X/C)$. So we can find [see Lemma 2.2 of Jas and Banerjee (1981)], τ-u.s.c and $\mathbf{V}$-l.s.c. function $g: X \to [0, 1]$ such that $g(D) = 0$ and $g(C) = 1$. Hence application of Theorem 2.1 completes the proof.

While searching for a unique q-proximity compatible with a given bitopological space we take special type of pairwise normal bitopological space. We prove the following theorem in this connection.

Theorom 2.4—Every pairwise compact and pairwise regular space $(X, \tau, \mathbf{V})$ has a unique compatible q-proximity $\prec$ given by $A \prec B$ iff $\mathbf{V}\text{-cl}(A) \cap \tau\text{-cl}(X \setminus B) = \emptyset$.

PROOF : Using Theorem 1.3 and 1.5 it follows that $(X, \tau, \mathbf{V})$ is pairwise normal and pairwise completely regular. So by Theorem 2.1 of Jas and Banerjee (1981), there is a compatible q-proximity $\prec_1$ on X defined by $A \prec_1 B$ iff there is a τ-l.s c.

and **V**-u.s.c. function $f : X \to [0, 1]$ such that $f(A) = 1$ and $f(X \setminus B) = 0$. We show that $\prec$ and $\prec_1$ are equivalent. Let $A \prec_1 B$ hold. Then we have from Lemma 2.1 of Jas and Banerjee (1981) **V**-cl $(A) \prec_1 B$, and so **V**-cl $(A) \prec_1 \tau$-int (B) and hence **V**-cl $(A) \subset \tau$-int (B) i e., **V**-cl $(A) \cap \tau$-cl $(X \setminus B) = \emptyset$ i.e., $A \prec B$. Conversely suppose $A \prec B$ holds. So **V**-cl $(A) \cap \tau$-cl $(X \setminus B) = \emptyset$. By pairwise normality of the space we find τ-l.s.c. and **V**-u.s.c. function $g : X \to [0, 1]$ satisfying $g(A) = 1$ and $g(X \setminus B) = 0$. This means $A \prec_1 B$.

Finally let $\prec_2$ be any q-proximity compatible with $(X, \tau, \mathbf{V})$. The proof will be complete upon showing that $\prec$ and $\prec_2$ are equivalent. Let $A \prec_2 B$, then we deduce as above that **V**-cl$(A) \cap \tau$-cl $(X \setminus B) = \emptyset$, and hence $A \prec B$ holds. Conversely suppose $A \prec B$ holds i.e., we have **V**-cl $(A) \cap \tau$-cl $(X \setminus B) = \emptyset$. Let $x \in$ **V**-cl (A); clearly $x \notin \tau$-cl $(X \setminus B)$. i.e., $x \in X \setminus \tau$-cl $\subset (X \setminus B) = \tau$-int $(B) = \tau_{\prec_2}$-int (B). So $\{x\} \prec_2 B$ holds. So there exists a subset C of X such that $\{x\} \prec_2 C \prec_2 B$. By Lemma 2.1 of Jas and Banerjee (1981), we have $\{x\}\ \alpha_{2\prec}\ \tau_{\prec_2}$-int $(C) \subset C \prec_2 B$. So there exists τ-open sets H_x such that $[H_x : x \in$ **V**-cl $(A)]$ is a τ-open cover of **V**-cl (A). Hence $[H_x : x \in$ **V**-cl $(A)] \cup (X \setminus$ **V**-cl $(A)) \subset \tau \cup \mathbf{V}$, is a pairwise open cover of X and thus has a finite subcover. Hence there exists a finite set of points $[x_1, x_2, \ldots, x_n]$ in X such that

$A \subset \bigcup_{k=1}^{n} H_{x_k}$, and $H_{x_k} \prec_2 B$, $K = 1,2,\ldots, n$. By Q4 we have $\bigcup_{k=1}^{n} H_{x_k} \prec_2 B$. So

$A \prec_2 B$. Hence $\prec$ and $\prec_2$ are equivalent.

Corollary 2.1—Every pairwise Hausdorff and pairwise compact bitopological space $(X, \tau, \mathbf{V})$ has a compatible unique q-proximity $\prec$ defined by $A \prec B$ iff **V**-cl $(A) \cap \tau$-cl $(X \setminus B) = \emptyset$.

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