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A METRIC FOR NON-ARCHIMEDEAN ENTIRE FUNCTIONS

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A metric for the space of non-Archimedean entire functions is defined.

Iyer (1948) has defined a metric for the space of entire functions, with the coefficients of the power series expansion of the functions. This metric works in the non-Archimedean case too. In this note, we define another metric with the zeros of the functions. It may be worth recording this latter metric for t, the space of non-Archimedean entire functions.

Let K be a non-Archimedean, non-trivial, complete, algebraically closed, and real valued field. By Weierstrass theorem [see Bruhat (1963), Proposition 3, p. 114], an entire function $f(z)$ over K can be written as

$$f(z) = Az^p \prod_{i=1}^{\infty} \left(1 - \frac{z}{a_i}\right)$$

where A is a constant, and $0 < | a_1 | \leqslant | a_2 | \leqslant \ldots \to \infty$. Let $z_0 \in K$. Set

$$S(z)_{z_0, t, f} = z^p \left(1 - \frac{z}{a_1}\right) \ldots \left(1 - \frac{z}{a_m}\right) \text{ if } | z_0 | \leqslant 1/t$$

$$\left(1 - \frac{z}{a_{m_1}}\right)\left(1 - \frac{z}{a_{m_1+1}}\right) \ldots \left(1 - \frac{z}{a_{m_1+l}}\right) \text{ if } | z_0 | > 1/t,$$

where $0, a_1, \ldots, a_m$ when $| z_0 | \leqslant 1/t$, and $a_{m_1}, a_{m_1+1}, \ldots, a_{m_1+l}$ when $| z_0 | > 1/t$ are the only zeros of $f(z)$ which satisfy $| z - z_0 | \leqslant 1/t$, $0 \leqslant t \leqslant \infty$. Let $r \in K$ be such that $\max(| z_0 |, 1/t) < | r |$. From corollary to Theorem 13 (see Adams 1966), we have, the number of zeros of $S(z)_{z_0, t, f}$ in $| z - z_0 | \leqslant 1/t$ is equal to

$$M(z_0, t, f) = \int_{0, r} \frac{z\, S'_{z_0, t, f}(z)}{S_{z_0, t, f}(z)}\, dz.$$

If

$$d(f, g) = \max\left(\sup_{\substack{z_0 \in K \\ t > 0}} | M(z_0, t, f) - M(z_0, t, g) |,\ | A - B |\right),$$

where $f(z) = A z^p \prod_{i=1}^{\infty} \left(1 - \frac{z}{a_i}\right) = A\Pi_f$

and $g(z) = B z^q \prod_{i=1}^{\infty} \left(1 - \frac{z}{b_i}\right) = B\,\Pi_g$

both belong to Γ, then we show that $d\,(\ldots)$ is actually the distance function on Γ.

Clearly, $0 \leqslant d(f, g) < \infty$, $d(f, g) < \max\,(d(f, h), d(h, g))$, $h(z) \in \Gamma$ (actually an ultrametric inequality is satisfied), $d(f, g) = d(g, f)$, and $d(f, g) = 0$ if $f = g$. So, we have only to show that $f \equiv g$, if $d(f, g) = 0$. Suppose, $f \not\equiv g$. Then, any one of the following $A = B$, $\Pi_f \not\equiv \Pi_g$; $A \neq B$, $\Pi_f \equiv \Pi_g$; $A \neq B$, $\Pi_f \not\equiv \Pi_g$ holds. Obviously, the second of these does not hold. If any one of the remaining two holds, then we can find z_0 and t such that

$0 < \max\,(\,|M(z_0, t, f) - M(z_0, t, g)|\,,\ |A - B|\ \leqslant d(f, g) = 0$, which is an impossibility. So, $f \equiv g$.

If

$$f_n(z) = A_n z^{p_n} \prod_{i=1}^{\infty} \left(1 - \frac{z}{a_{n,i}}\right),$$

$n = 1, 2, \ldots$ is a Cauchy sequence of Γ, then for a given $\epsilon > 0$, we get $d(f_{n+p}, f_n) < \epsilon$ for $n \geqslant n_0$, $p \geqslant 1$. Let $A_n \to A$, $p_n \to p_0$, and $a_{n,i} \to a_i$ as $n \to \infty$. If

$$f(z) = A z^{p_0} \prod_{i=1}^{\infty} \left(1 - \frac{z}{a_i}\right),$$

then $d(f_n, f) < \epsilon$ for $n \geqslant n_0$. So, from Weierstrass theorem, it follows that $f(z)$ is an entire function. As in, Von Rooij and Schikhof (1971, p. 130), it follows that Γ is zero dimensional. Clearly it is Hausdroff. Thus, it is a complete and totally disconnected metric space.

So, we have the following :

Theorem 1—Γ is a complete and totally disconnected metric space.

One can prove results that were proved by Iyer and others for this space too—but the non-Archimedean analogues in these cases follow by essentially same arguments!

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