Indian J. pure appl. Math., 13(10) : 1149–1162, October 1982

AN INTEGRAL EQUATION INVOLVING FOX'S *H*-FUNCTION

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(*Received* 12 *October* 1981)

Fox's function can be considered a generalization of the Bessel function of the first kind, and hence useful in many practical applications. In this paper we consider a singular integral equation with Fox's function as a kernel and develop methods to invert it. The technique consists of applying successively, linear differential operators of infinite order of the type $\Gamma(\alpha+\beta\theta)$, and operators of the type $\frac{1}{\Gamma(\alpha+\beta\theta)}$, where Γ is the Euler's gamma function and $\theta = -x \frac{d}{dx}$.

1. INTRODUCTION

Fox (1965) introduced a function called *H*-function of order *n* defined as

$$H\left(x \left| \begin{matrix} \alpha_i, a_i \\ \beta_i, a_i \end{matrix} ; n \right.\right) = H\left(x \left| \begin{matrix} \alpha_1, a_1; \alpha_2, a_2; \ldots; \alpha_n, a_n \\ \beta_1, a_1; \beta_2, a_2; \ldots; \beta_n, a_n \end{matrix} \right.\right)$$

$$= \frac{1}{2\pi i} \int_C \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + s a_i)}{\Gamma(\beta_i - s a_i)} x^{-s} \, ds \qquad \ldots(1.1)$$

with the following conditions :

(i) a_i, α_i, β_i are all real $i = 1, 2 \ldots, n$.

(ii) $a_i > 0$

(iii) Let $s = \sigma + i\tau$, where σ and τ are real; then the contour C along with the integral of (1.1) is taken as the straight line whose equation is $\sigma = \sigma_0$, where σ_0 is a constant. This line is parallel to the imaginary axis in the complex s-plane.

(iv) All the poles of the integrand of (1.1) are simple and lie on the left of the line $\sigma = \sigma_0$. This requires $\sigma_0 > -\frac{\alpha_i}{a_i}$, $i = 1, 2 \ldots, n$.

(va) $2\sigma_0 \sum_{i=1}^{n} a_i < \sum_{i=1}^{n} (\beta_i - \alpha_i)$

(vb) $2\sigma_0 \sum_{i=1}^{n} a_i < \sum_{i=1}^{n} (\beta_i - \alpha_i) - 1.$

It is easy to see that using the asymptotic expansion of the Γ-functions, the integral of (1 1) exists if (Va) holds and is absolutely convergent if (Vb) holds.

The expansion also shows that the contour C of (1.1) can be closed by a large semicircle on the left. Calculating the residues, one can find that the H-function of order n can be expressed as the sum of n power series, the ith of which is multiplied by x^{α_i/a_i}. Each of these series is an entire function. In particular, H-function of order 1 is

$$H\left(x \,\middle|\, \begin{matrix}\alpha, \frac{1}{2}\\ \beta, \frac{1}{2}\end{matrix}\,;1\right) = 2\,x^{\alpha-\beta+1}\,J_{\alpha+\beta-1}(2x)$$

where J denotes the Bessel function.

The integral equation we consider here is

$$f(x) = \int_0^\infty H\left(xt \,\middle|\, \begin{matrix}\alpha_i, a_i\\ \beta_i, a_i\end{matrix}\,;n\right) g(t)\,dt,\; x > 0 \qquad \ldots(1.2)$$

where $f(x)$ is given and $g(x)$ is to be determined. We assume that the H-function satisfies the above five conditions. Our aim in this paper, is to invert the equation (1.2). We shall develop certain operators which when applied to the function $f(x)$ shall yield the unknown function $g(x)$, establishing the inversion (Nasim 1975, Widder 1971). This inversion operator is in fact composition of integral and differential operators, to be defined in the next section.

2. The Operators

It is an easy matter to see that the operator θ^n, where $\theta = -x\dfrac{d}{dx}$ and n a positive integer has the property that

$$\theta^n[x^s] = (-s)^n x^s,$$

for some constant number s. Hence a polynomial $p_n(\theta)$ of order n in θ, gives

$$p_n(\theta)[x^s] = p_n(-s)\,x^s.$$

Similarly

$$p(\theta)[x^s] = \lim_{n\to\infty} p_n(\theta)[x^s]$$

$$= \lim_{n\to\infty} p_n(-s)\,x^s$$

$$= p(-s)\,x^s.$$

Note that the effect of linear differential operator of order n or linear differential operator of infinite order on a function x^s is to reproduce it with a constant multiplier.

Now the operator $n^{-\theta}$, can be expressed symbollically as

$$n^{-\theta} = e^{-\theta \ln n} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-\ln n)^k}{k!}\,\theta^k$$

$$= \lim_{N\to\infty} p_N(\theta), \text{ say.}$$

Thus,

$$n^{-\theta}[x^s] = n^s x^s, \text{ for some } s. \qquad \text{...(2.1)}$$

With this understanding and using the Euler product for the gamma function, we can write symbollically,

$$\frac{1}{\Gamma(1+\theta)} = \lim_{n\to\infty} n^{-\theta} \prod_{k=1}^{n} \left(1+\frac{\theta}{k}\right)$$

$$= \lim_{n\to\infty} n^{-\theta} p_n(\theta). \qquad \text{...(2.2)}$$

This is a known operator of the type introduced in (Widder 1971, Chapter IX). Once again from (2.1) and (2.2), we have

$$\frac{1}{\Gamma(1+\theta)}[x^s] = \frac{1}{\Gamma(1-s)} x^2.$$

More generally,

$$\frac{1}{\Gamma(\alpha+a\theta)}[x^s] = \frac{1}{\Gamma(\alpha-as)} x^s \qquad \text{...(2.3)}$$

where α, a are real and s a complex constant. We shall call the operator $\frac{1}{\Gamma(\alpha+a\theta)}$, a linear differential operator of infinite order. Next we shall consider operators of the type denoted by $\Gamma(\beta - a\theta)$, where both β and a are real, $a > 0$ and θ as usual denotes the operator $-x\frac{d}{dx}$. We define the operator $\Gamma(1-\theta)$ by the equation

$$\Gamma(1-\theta)[f(x)] = \int_0^\infty e^{-t} f(xt)\, dt$$

and more generally

$$\Gamma(\beta - a\theta)[f(x)] = \int_0^\infty k(t) f(xt)\, dt$$

where

$$k(x) = \frac{1}{a} x^{\beta/\alpha - 1} e^{-x^{1/a}}, \; a > 0.$$

Or,

$$\Gamma(\beta - a\theta)[f(x)] = \int_0^\infty f(xt^a)\, t^{\beta-1} e^{-t}\, dt. \qquad \text{...(2.4)}$$

If $f(x) = x^s$, then

$$\Gamma(\beta - a\theta)[x^s] = x^s \int_0^\infty t^{as+\beta-1} e^{-t}\, dt$$

$$= \Gamma(\beta + sa)\, x^s, \; R(\beta + sa) > 0.$$

We note that the effect of the operator $\Gamma(\beta - a\theta)$ when applied to the function x^s, is to simply reproduce it with a constant multiplier. So, although the operator $\Gamma(\beta - a\theta)$ is an integral operator, defined by eqn. (2.4), it behaves like a differential operator in the sense of its effect on the function x^s. Now, formally, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

then

$$\Gamma(\beta - a\theta)[f(x)] = \sum_{n=0}^{\infty} a_n \Gamma(\beta + an) x^n, \; \beta + an > 0$$

for some suitable interval of convergence. We give a few examples to illustrate the behaviour of the operator $\Gamma(\beta - a\theta)$.

Let $f(x) = x^\mu J_\nu(x)$ and $a = \frac{1}{2}$.

Then due to (2.4),

$$\Gamma(\beta - \tfrac{1}{2}\theta)[x^\mu J_\nu(x)] = x^\mu \int_0^\infty J(xt^{1/2})\, t^{\beta+(1/2_\nu)\mu-1} e^{-t}\, dt$$

$$= \frac{\Gamma(\beta + \frac{1}{2}\mu + \frac{1}{2}\nu)}{\Gamma(\nu+1)\, 2^\nu} x^{\mu+\nu} {}_1F_1(\beta + \tfrac{1}{2}\mu + \tfrac{1}{2}\nu; \nu + 1; -\frac{x^2}{4}) \qquad \ldots(2.5)$$

where $\beta + \frac{1}{2}\mu + \frac{1}{2}\nu > 0$ (Erdélyi *et al.* 1954a, p. 186).

On the other hand,

$$\Gamma(\beta - \tfrac{1}{2}\theta)[x^\mu J_\nu(x)] = \Gamma(\beta - \tfrac{1}{2}\theta)\; x^\mu \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} (\tfrac{1}{2})^{2n+\nu}\, \Gamma(\beta - \tfrac{1}{2}\theta)[x^{2n+\mu+\nu}]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} (\tfrac{1}{2})^{2n+\nu}\, \Gamma(\beta + n + \tfrac{1}{2}\mu + \tfrac{1}{2}\nu)\, x^{9n+\nu+\mu}$$

$$= \frac{\Gamma(\beta + \frac{1}{2}\mu + \frac{1}{2}\nu)}{\Gamma(\nu+1)\, 2^\nu} {}_1F_1(\beta + \tfrac{1}{2}\mu + \tfrac{1}{2}\nu; \nu + 1; -\frac{x^2}{4})$$

which agrees with (2.5), deduced above.

Also if we let

$f(x) = {}_mF_n(a_1, \ldots, a_m; \rho_1 \ldots, \rho_n; x)$, then

using the definition (2.4),

$$\Gamma(\beta - a\theta)[f(x)] = \Gamma(\beta)\, {}_{m+a}F_n\left(a_1, \ldots, a_m, \frac{\beta}{a}, \frac{\beta+1}{a}, \ldots, \frac{\beta+a-1}{a}; \rho_1, \ldots, \rho_n; x\, a^a\right) \qquad \ldots(2.6)$$

where a is a positive integer, $m + a \leqslant n$, and $\beta > 0$ Erdélyi *et al.* (1954a, p. 220) whereas

$$\Gamma(\beta - a\theta)[f(x)] = \Gamma(\beta - a\theta)\left[\sum_{k=0}^{\infty} \frac{(a_1)_k \quad (a_m)_k\, x^k}{(\rho_1)_k \dots (\rho_n)_k\, k!}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_m)_k}{(\rho_1)_k \dots (\rho_n)_k\, k!}\, \Gamma(\beta - a\theta)[x^k]$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_m)_k\, \Gamma(\beta + ak)}{(\rho_1)_k \dots (\rho_n)_k\, k!}\, x^k . \qquad \dots(2.7)$$

Now by Gauss multiplication theorems (Whittaker and Watson 1963, p. 240)

$$\Gamma(\beta + ak) = \Gamma\left(a\left(\frac{\beta}{a} + k\right)\right] = a^{\beta + ak - 1/2}(2\pi)^{1/2\,(1-a)}\,\Gamma\left(\frac{\beta}{a} + k\right)$$
$$\times \Gamma\left(\frac{\beta + 1}{a} + k\right) \dots \Gamma\left(\frac{\beta + a - 1}{a} + k\right)$$

and

$$\Gamma(\beta) = \Gamma\left(a\left(\frac{\beta}{a}\right)\right] = a^{\beta - 1/2}(2\pi)^{1/2(1-a)}\,\Gamma\left(\frac{\beta}{a}\right)\Gamma\left(\frac{\beta + 1}{a}\right) \dots \Gamma\left(\frac{\beta + a - 1}{a}\right).$$

Substituting these products for $\Gamma(\beta + ak)$ and $\Gamma(\beta)$ in (2.7) and simlifying, we have

$$\Gamma(\beta - a\theta)\,[{}_mF_n(a_1, \dots, a_m; \rho_1, \dots, \rho_n; x)]$$

$$= \Gamma(\beta) \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_m)_k \left(\frac{\beta}{a}\right)_k \dots \left(\frac{\beta + 1}{a}\right)_k (x\, a^a)^k}{(\rho_i)_k \dots (\rho_m)_k\, k!}$$

$$= \Gamma(\beta)\, {}_{m+a,n}F\left(a_1, \dots, a_m, \frac{\beta}{a}, \frac{\beta + 1}{a}, \dots, \frac{\beta + a - 1}{a}; \rho_1, \dots, \rho_n; xa^a\right)$$

as deduced in (2.7). This established the algorithm for the operator $\Gamma(\beta - a\theta)$.

This leads us to define the operator

$$L(\theta) = \prod_{i=1}^{n} \frac{\Gamma(\beta_i - a_i\theta)}{\Gamma(\alpha_i + a_i\theta)} \qquad \dots(2.8)$$

where a_i, α_i, β_i are all real and $a_i > 0$. This operator is composition of the two types of operators defined above by eqns. (2.2) and (2.4). It is now an easy matter to see that

$$L(\theta)[x^{-s}] = \prod_{i=1}^{n} \frac{\Gamma(\beta_i - a_i\theta)}{\Gamma(\alpha_i + a_i\theta)}[x^{-s}]$$

(*equation continued on p.* 1154)

$$= \prod_{i=1}^{n} \frac{\Gamma(\beta_i - a_i s)}{\Gamma(\alpha_i + a_i s)} x^{-s} \qquad \ldots(2.9)$$

for some $s = \sigma + i\tau$ and $(\beta_i - a_i s) > 0$, $i = 1, 2, \ldots, n$.

3. The Inversion

We shall make use of the Mellin transform theory and therefore some definitions are in order. We denote the Mellin transform of $f(x)$ by $F(s)$, then

$$F(s) = \int_0^\infty f(x) x^{s-1} dx \qquad \ldots(3.1)$$

and

$$f(x) = \frac{1}{2\pi i} \int_C F(s) x^{-s} ds \qquad \ldots(3.2)$$

where the contour C is usually a straight line parallel to the imaginary axis in the complex $s = \sigma + i\tau$ plane, with $\sigma = \sigma_0$, a constant. If $F(s)$ and $G(s)$ denote the Mellin transforms of $f(x)$ and $g(x)$ respectively, then the Parseval theorem states that

$$\int_0^\infty f(xt) g(t) dt = \frac{1}{2\pi i} \int_C F(s) G(1 - s) x^{-s} ds. \qquad \ldots(3.3)$$

The condition of validity of (3.1), (3.2) and (3.3) can be found in [Titchmarsh 1948, §§ 1.29, 3. 17, 4. 14], respectively. But for convenience, we shall assume that the function $f(x)$ and $g(x)$ both belong to $L^2(0, \infty)$. Then $F(s)$ and $G(s)$ also both belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and the contour C is the line $\sigma_0 = \frac{1}{2}$. Now from the definition of the H-function given by (1.1). we infer that

$$\prod_{i=1}^{n} \frac{\Gamma(\alpha_i + a_i s)}{\Gamma(\beta_i - a_i s)}, \; s = \sigma + i\tau, \; -\infty < \tau < \infty$$

is the Mellin transform of $H\left(x \,\middle|\, \begin{matrix} \alpha_i, a_i \\ \beta_i, a_i \end{matrix} ; n\right)$ and belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, if we assume the condition (Vb) and consequently $H\left(x \,\middle|\, \begin{matrix} \alpha_i, a_i \\ \beta_i, a_i \end{matrix} ; n\right)$ belongs to $L^2(0, \infty)$.

The Main Theorem

Let $g(x) \in L^2(0, \infty)$.

If $$f(x) = \int_0^\infty H\left(xt \,\middle|\, \begin{matrix} \alpha_i, a_i \\ \beta_i, a_i \end{matrix} ; n \right) g(t) dt \qquad \ldots(3.4)$$

then, $L(\theta)[f(x)] = \frac{1}{x} g\left(\frac{1}{x}\right); a \cdot a \cdot x > 0$ ...(3.5)

where the operator $L(\theta)$ is defined by (2.8).

PROOF : It is clear that the integral in the eqn. (3.4) is absolutely convergent since both the function H and g belong to $L^2(0, \infty)$. Consequently, the Mellin transforms of these functions also belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence by the Parseval theorem (3.3), and from (3.4)

$$f(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + a_i s)}{\Gamma(\beta_i - a_i s)} G(1-s) x^{-s} ds$$

Now,

$$L(\theta)[f(x)] = L(\theta)\left[\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + a_i s)}{\Gamma(\beta_i - a_i s)} G(1-s) x^{-s} ds\right]$$

$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + a_i s)}{\Gamma(\beta_i - a_i s)} G(1-s)\, ds\, L(\theta)[x^{-s}]$$

$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + a_i s)}{\Gamma(\beta_i - a_i s)} \prod_{i=1}^{n} \frac{\Gamma(\beta_i - a_i s)}{\Gamma(\alpha_i + a_i s)} G(1-s) x^{-s} ds,$$

$$= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} G(1-s) x^{-s} ds$$

$$= \frac{1}{x} g\left(\frac{1}{x}\right) a \cdot a \cdot x > 0,$$

using the fact that $G(1-s)$ is the Mellin transform of $\frac{1}{x} g\left(\frac{1}{x}\right)$ if $G(s)$ denotes the Mellin transform of $g(x)$ and using the property (2.9) of the operator $L(\theta)$.

This establishes the result, provided we can justify bringing the operator $L(\theta)$ inside the integral sign in the analysis above. The operator $L(\theta)$ is composed of operators of the type defined in (2.2) and (2.4), hence the application of the operator $L(\theta)$ inside the integral sign amounts to successively integrating and differentiating inside the integral which can be justified by the classical results since the resulting integral is uniformly convergent on every compact set of the x-axis.

Although the interpretation of the operator $L(\theta)$ is straightforward enough, as seen in the previous section, its application is not. We give below, a few examples to illustrate the different cases which arise in the course of the applications of the operator $L(\theta)$. We shall consider special cases of the H-function, in the examples.

Example 1 — Let $\alpha = \frac{1}{2}$, $\beta = 1$, $a = \frac{1}{2}$ and $n = 1$.

Then, $$H\left(x \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1, \frac{1}{2}, \end{matrix} ; 1\right) = 2\,\pi^{-1/2} \sin(2x)$$

and the corresponding operator is

$$L(\theta) = \frac{\Gamma(1 - \frac{1}{2}\theta)}{\Gamma(\frac{1}{2} + \frac{1}{2}\theta)} .$$

Now if, $g(x) = e^{-x}$ then

$$f(x) = 2\,\pi^{-1/2} \int_0^\infty \sin(2xt)\, e^{-t}\, dt$$

$$= 2\,\pi^{-1/2} \frac{x}{1+4x^2}, \text{ (Erdélyi } et\ al. \text{ 1954a, p. 150)}$$

$$= \pi^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} x^{-2n-1}, \; |2x| > 1$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(1+\frac{1}{2}k)}{k!\, \Gamma(\frac{1}{2}-\frac{1}{2}k)} x^{-k-1}.$$

Thus,

$$L(\theta)[f(x)] = \frac{\Gamma(1-\frac{1}{2}\theta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\theta)} \left[\sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(1+\frac{1}{2}k)}{k!\, \Gamma(\frac{1}{2}-\frac{1}{2}k)} x^{-k-1} \right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(1+\frac{1}{2}k)}{k!\, \Gamma(\frac{1}{2}-\frac{1}{2}k)} \frac{\Gamma(1-\frac{1}{2}\theta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\theta)} [x^{-k-1}]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(1+\frac{1}{2}k)\, \Gamma(\frac{1}{2}-\frac{1}{2}k)}{k!\, \Gamma(\frac{1}{2}-\frac{1}{2}k)\, \Gamma(1+\frac{1}{2}k)} x^{-k-1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{-k-1}$$

$$= \frac{1}{x} e^{-1/x}$$

$$= \frac{1}{x} g\left(\frac{1}{x}\right),$$

whence,

$$g(x) = e^{-x}$$

as predicted.

Example 2 — If $a = \frac{1}{2}$ and $n = 1$, then

$$H\left(x\left|\begin{matrix}\alpha, \frac{1}{2}\\ \beta, \frac{1}{2}\end{matrix}\right.; 1\right) = 2x^{\alpha+\beta-1} J_{\alpha+\beta-1}(2x)$$

and $$L(\theta) = \frac{\Gamma(\beta-\frac{1}{2}\theta)}{\Gamma(\alpha+\frac{1}{2}\theta)}.$$

Now let

$$g(x) = x^{\mu-1/2} e^{-x^2}$$

then, (Erdélyi *et al.* 1954a, p. 186)

$$f(x) = 2x^{\alpha-\beta+1} \int_0^\infty J_{\alpha+\beta-1}(2xt)\, t^{\mu+\alpha-\beta+1} e^{-t^2} dt$$

$$= \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4})}{\Gamma(\alpha+\beta)} x^{2\alpha} {}_1F_1(\alpha+\tfrac{1}{2}\mu+\tfrac{1}{4}; \alpha+\beta; -x^2),$$

$$\alpha+\tfrac{1}{2}\mu+\tfrac{1}{4} > 0, \alpha+\beta > 0$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4}+n)(-1)^n}{\Gamma(\alpha+\beta+n)\, n!} x^{2n+2\alpha}, \ x > 0.$$

We first consider

$$\Gamma(\beta-\tfrac{1}{2}\theta)[f(x)] = \Gamma(\beta-\tfrac{1}{2}\theta)\left[\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4}+n)(-1)^n}{\Gamma(\alpha+\beta+n)\, n!} x^{2n+2\alpha}\right],$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4}+n)(-1)^n}{\Gamma(\alpha+\beta+n)\, n!} \Gamma(\beta-\tfrac{1}{2}\theta)[x^{2n+2\alpha}]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4}+n)(-1)^n}{\Gamma(\alpha+\beta+n)\, n!} \Gamma(\beta+n+\alpha)\, x^{2n+2\alpha}$$

$$= \Gamma(\alpha+\tfrac{1}{2}\mu+\tfrac{1}{4})\, x^{2\alpha}(1+x^2)^{-(4\alpha+2\mu+1)/4}, \ |x| < 1$$

$$= \Gamma(\alpha+\tfrac{1}{2}\mu+\tfrac{1}{4})\, x^{-\mu-1/2}\left(1+\frac{1}{x^2}\right)^{-(4\alpha+2\mu+1)/4}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4}+n)(-1)^n}{n!} x^{-(4n+2\mu+1)/2} \ |x| > 1.$$

Then,

$$\frac{1}{\Gamma(\alpha+\frac{1}{2}\theta)}[\Gamma(\beta-\tfrac{1}{2}\theta) f(x)]$$

$$= \frac{1}{\Gamma(\alpha+\frac{1}{2}\theta)}\left[\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4}+n)(-1)^n}{n!} x^{-(4n+2\mu+1)/2}\right]$$

(equation continued on p. 1158)

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{4}+n)(-1)^n}{n!} \frac{1}{\Gamma(\alpha+\frac{1}{2}\theta)} [x^{-(4n+2\mu+1)/2}]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{-(4n+2\mu+1)/2}$$

$$= x^{-\mu-1/2} e^{-1/x^2}$$

$$= \frac{1}{x} g\left(\frac{1}{x}\right),$$

whence,

$$g(x) = x^{\mu-1/2} e^{-x^2}, \; x > 0$$

as required.

Example 3 — Again let

$$H\left(x \,\middle|\, \begin{matrix}\alpha, \frac{1}{2}\\ \beta, \frac{1}{2}\end{matrix}; 1\right) = 2\, x^{\alpha-\beta+1} J_{\alpha+\beta-1}(2x),$$

and

$$L(\theta) = \frac{\Gamma(\beta-\frac{1}{2}\theta)}{\Gamma(\alpha+\frac{1}{2}\theta)}.$$

If $$g(x) = \begin{cases} x^{\mu}(a^2-x^2)^{\lambda}, & 0 < x < a \\ 0 & a < x < \infty \end{cases}$$

then (Erdélyi *et al.* 1954b, p. 26),

$$f(x) = \int_0^{\infty} H\left(xt \,\middle|\, \begin{matrix}\alpha, \frac{1}{2}\\ \beta, \frac{1}{2}\end{matrix}; 1\right) g(t)\, dt$$

$$= \frac{a^{2\lambda+2\alpha+\mu+1} B(\lambda+1, \alpha+\frac{1}{2}\mu+\frac{1}{2})}{\Gamma(\alpha+\beta)} x^{2\alpha}$$

$$\times\, {}_1F_2(\alpha+\tfrac{1}{2}\mu+\tfrac{1}{2}; \alpha+\beta, \alpha+\tfrac{1}{2}\mu+\tfrac{3}{2}+\lambda; -a^2x^2)$$

$$= a^{2\lambda+2\alpha+\mu+1}\Gamma(\lambda+1)\, x^{2\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{2}+n)(-a^2x^2)^n}{\Gamma(\alpha+\beta+n)\Gamma(\alpha+\frac{1}{2}\mu+\frac{3}{2}+\lambda+n)\, n!}.$$

Now

$$\Gamma(\beta-\tfrac{1}{2}\theta)[f(x)] = a^{2\lambda+2\alpha+\mu+1}\Gamma(\lambda+1) \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{2}+n)(-a^2)^n}{\Gamma(\alpha+\beta+n)\Gamma(\alpha+\frac{1}{2}\mu+\frac{3}{2}+\lambda+n)\, n!}$$

$$\times\; \Gamma(\beta-\tfrac{1}{2}\theta)[x^{2n+2\alpha}]$$

(*equation continued on p* 1159)

$$= \frac{a^{2\lambda+2\alpha+\mu+1}\,\Gamma(\lambda+1)\,\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2}\mu+\frac{3}{2}+\lambda)}\,x^{2\alpha}$$

$$\times\ {}_1F_1(\alpha+\tfrac{1}{2}\mu+\tfrac{1}{2};\ \alpha+\tfrac{1}{2}\mu+\tfrac{3}{2}+\lambda;\ -a^2x^2)$$

$$= a^{2\lambda+2\alpha+\mu+1}\,\Gamma(\lambda+1)\,x^{2\alpha}\,G^{1,1}_{2,1}\left(\frac{1}{a^2x^2}\,\middle|\,\begin{matrix}1,\ \alpha+\frac{1}{2}\mu+\frac{3}{2}+\lambda\\ \alpha+\frac{1}{2}\mu+\frac{1}{2}\end{matrix}\right)$$

$$= a^{2\lambda+2\alpha+\mu+1}\,\Gamma(\lambda+1)\,x^{2\alpha}\,\frac{1}{2\pi i}\int_C\frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{2}-s)\,\Gamma(s)}{\Gamma(\alpha+\frac{1}{2}\mu+\frac{3}{2}+\lambda-s)}\left(\frac{1}{a^2x^2}\right)^s ds.$$

Therefore,

$$\frac{1}{\Gamma(\alpha+\frac{1}{2}\theta)}\,[\Gamma(\beta-\tfrac{1}{2}\theta)\,f(x)] = a^{2\lambda+2\alpha+\mu+1}\,\Gamma(\lambda+1)\,x^{2\alpha}$$

$$\times\frac{1}{2\pi i}\int_C\frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{2}-s)}{\Gamma(\alpha+\frac{1}{2}\mu+\frac{3}{2}+\lambda-s)}\,(a\,x)^{-2s}\,ds$$

by applying the operator inside the integral sign and simplifying

$$= \frac{1}{x}\,g\left(\frac{1}{x}\right),$$

due to our main theorem. Hence

$$g(x) = a^{2\lambda+2\alpha+\mu+1}\,\Gamma(\lambda+1)\,x^{-2\alpha-1}\,2\pi i\int_C\frac{\Gamma(\alpha+\frac{1}{2}\mu+\frac{1}{2}-s)}{\Gamma(\alpha+\frac{1}{2}\mu+\frac{3}{2}+\lambda-s)}\left(\frac{x}{a}\right)^{2s} ds.$$

Now letting $\frac{a^2}{2} = t$, and making use of the Mellin transform theory, we obtain (Erdélyi 1954a, p. 350)

$$g(x) = a^{2\lambda+\mu}\,t^{\alpha+1/2}\begin{cases}0, & 0<t<1\\ t^{-(2\alpha+\mu+1+2\lambda)/2} & (t-1)^{\lambda},\ t>1.\end{cases}$$

Hence, on substituting back for t and simplifying,

$$g(x) = \begin{cases}x^{\mu}(a^2-x^2)^{\lambda}, & 0<x<a\\ 0, & a<x<\infty\end{cases}$$

as predicted.

Example 4 — Consider once again

$$H\left(x\,\middle|\,\begin{matrix}\alpha,\ \frac{1}{2}\\ \beta,\ \frac{1}{2}\end{matrix};\ 1\right) = 2\,x^{\alpha-\beta+1}\,J_{\alpha+\beta-1}(2x)$$

and

$$L(\theta) = \frac{\Gamma(\beta-\frac{1}{2}\theta)}{\Gamma(\alpha+\frac{1}{2}\theta)}.$$

Let $g(x) = x^{\mu-2}\,e^{-ax}$, then (Erdélyi *et al.* 1954b, p. 29),

$$f(x) = \frac{2\Gamma(2\alpha+\mu+1)}{a^{2\alpha+\mu-1}\Gamma(\alpha+\beta)} x^{2\alpha} {}_2F_1\left(\alpha+\tfrac{1}{2}\mu-\tfrac{1}{2}, \alpha+\tfrac{1}{2}\mu : \alpha+\beta; -\frac{4x^2}{a^2}\right),$$

$$2\alpha+\mu+1>0,\ a>0.$$

$$= \pi^{-1/2}\left(\frac{2}{a}\right)^{2\alpha+\mu-1} x^{2\alpha} G^{2,1}_{2,2}\left(\frac{a^2}{4x^2}\,\middle|\,\begin{matrix}1, \alpha+\beta\\ \alpha+\frac{1}{2}\mu-\frac{1}{2}, \alpha+\frac{1}{2}\mu\end{matrix}\right)$$

$$= \pi^{-1/2}\left(\frac{2}{a}\right)^{2\alpha+\mu-1} x^{2\alpha}\frac{1}{2\pi i}\int_C \frac{\Gamma(\alpha+\frac{1}{2}\mu-\frac{1}{2}-s)\Gamma(\alpha+\frac{1}{2}\mu-s)\Gamma(s)}{\Gamma(\alpha+\beta-s)}\left(\frac{a^2}{4x^2}\right)^s ds$$

Now,

$$\frac{\Gamma(\beta-\frac{1}{2}\theta)}{\Gamma(\alpha+\frac{1}{2}\theta)}[f(x)] = \pi^{-1/2}\left(\frac{2}{a}\right)^{2\alpha+\mu-1}\frac{1}{2\pi i}$$

$$\int_C \frac{\Gamma(\alpha+\frac{1}{2}\mu-\frac{1}{2}-s)\Gamma(\alpha+\frac{1}{2}\mu-s)\Gamma(s)}{\Gamma(\alpha+\beta-s)}\left(\frac{a^2}{4}\right)^s$$

$$\times\frac{\Gamma(\beta-\frac{1}{2}\theta)}{\Gamma(\alpha+\frac{1}{2}\theta)}[x^{2\alpha-2s}]\, ds$$

$$= \pi^{-1/2}\left(\frac{2}{a}\right)^{2\alpha+\mu-1}\frac{1}{2\pi i}\int_C \Gamma(\alpha+\tfrac{1}{2}\mu-\tfrac{1}{2}-s)\Gamma(\alpha+\tfrac{1}{2}\mu-s)\left(\frac{a^2}{4}\right)^s x^{2\alpha-2s}\, ds$$

$$= a^{-(2\alpha+\mu-1)} x^{2\alpha}\frac{1}{2\pi i}\int_C \Gamma(2\alpha+\mu-1-2s)\left(\frac{a}{x}\right)^{2s} ds.$$

The integrand in the above integral has poles at $2s = 2\alpha+\mu-1+n$. By the residue theory, the right-hand side gives

$$= a^{-(2\alpha+\mu-1)} x^{2\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{a}{x}\right)^{2\alpha+\mu-1+n}$$

$$= x^{1-\mu} e^{-a/x}$$

$$= \frac{1}{x} g\left(\frac{1}{x}\right)$$

due to our main theorem. Hence,

$$g(x) = x^{-\mu 2} e^{-ax}$$

as desired.

To make the formula (3.5) feasible, we may consider the following inversion algorithms.

Case 1 — If

$$f(x) = \sum_{k=0}^{\infty} a_k x^{-k-\alpha}, \quad x > r \text{ for some } r$$

and α, then

$$g(x) = \sum_{k=0}^{\infty} a_k \prod_{i=1}^{n} \frac{\Gamma[\beta_i - a_i(k+\alpha)]}{\Gamma[\alpha_i + a_i(k+\alpha)]} x^{k+\alpha-1}, \; x > R$$

for some R and $a_i > 0$.

It is obvious that the region of convergence of the series for $g(x)$ is larger than the region of convergence of the series for $f(x)$, due to the multiplier $\prod_{i=1}^{n} \frac{\Gamma[\beta_i - a_i(k+\alpha)]}{\Gamma[\alpha_i + a_i(k+\alpha)]}$ in the series.

Case 2 — If

$$f(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}. \qquad x > r \text{ for some } r \text{ and } \alpha; \text{ and}$$

$$R(x) = \prod_{i=1}^{n} (\beta_i - a_i n) \; [f(x)] = \sum_{k=0}^{\infty} a_k x^{-k-\beta},$$

for some β, then

$$g(x) = \sum_{k=0}^{\infty} \frac{b_k}{\prod_{i=0}^{n} [\alpha_i + a_i(k+\beta)]} x^{k+\beta-1}.$$

Case 3 — If

$$f(x) = \frac{1}{2\pi i} \int_C F(s) x^{-s} ds, \; s = \sigma + i\tau, \; -\infty < \tau < \infty$$

and C is appropriately defined in the complex s-plane, then

$$g(x) = \frac{1}{2\pi i} \int_C \prod_{i=1}^{n} \frac{\Gamma(\beta_i - a_i s)}{\Gamma(\alpha_i + a_i s)} F(x) x^{s-1} ds,$$

where C' is also suitably defined.

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