Indian J. pure appl. Math., **13**(10) : 1163–1166, October 1982

TRANSFORMATION OF KAMPÉ DE FÉRIET FUNCTION

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(*Received* 3 *September* 1981)

In this paper, making use of fractional derivatives, a transformation for Kampé de Fériet function has been established from which certain formulae for the evaluation of double and triple series have been deduced.

1. INTRODUCTION

Recently, Srivastava (1971), Singal (1970) and Carlitz (1967) gave a number of transformations of Kampé de Fériet functions. In this paper an attempt has been made to establish a transformation involving Kampé de Fériet function by making use of some known results regarding fractional derivatives for the product of two-functions due to Manocha and Sharma (1974). In the sequel, interesting special cases have also been studied. Some of the results obtained as special cases are believed to be new.

For the sake of brevity, we shall use the modified notation of Singal (1970) to represent Kampé de Fériet function, viz.,

$$F^{p,q}_{r,s}\left[\begin{matrix} a_p : b_q ; b'_q ; \\ c_r : d_s ; d'_s ; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a_p)_{m+n} (b_q)_m (b'_q)_n x^m y^n}{(1)_m (1)_n (c_r)_{m+n} (d_s)_m (d'_s)_n} \qquad \ldots(1.1)$$

where $p + q \leqslant 1 + r + s$, a_p stands for the sequence of parameters $a_1, a_2, \ldots, a_p$ and $(a)_n = \dfrac{\Gamma[a+n]}{\Gamma[a]}$.

Following Manocha and Sharma (1974), we define the fractional derivative for the product of two functions as, follows :

If u and v are two functions of the form $x^{\alpha-1} f(x)$ and $x^{\beta} g(x)$ which are analytic in a circular region $|x| < \rho$, then in a suitable region of convergence

$$D_x^{\lambda} [u\, v] = \sum_{n=0}^{\infty} \binom{\lambda}{n} D_x^{n} (u)\; D_x^{\lambda-n} (v) \qquad \ldots(1.2)$$

where $\text{Re}(\alpha) > 0$, $\text{Re}(\alpha + \beta) > 0$ and $\text{Re}(\lambda) < 0$.

Also,

$$D_x^{\lambda}\,[x^{\mu-1}] = \frac{\Gamma[\mu]}{\Gamma[\mu-\lambda]}\,x^{\mu-\lambda-1} \qquad \ldots(1.3)$$

where $\lambda \neq \mu$.

2. Result

In this section we shall establish following result.

$$\sum_{t=0}^{\lambda} \frac{(-\lambda)_t\,(1-\alpha)_t}{(1)_t\,(\beta)_t}\; F^{p+1,q}_{r+1,s}\left[\begin{matrix} a_p,\ \beta+\lambda : b_q\,;\, b'_q\,; \\ c_r,\ \beta+t : d_s;\, d'_s\,; \end{matrix}\ b,\ \mu_1,\ x,\ \mu_2\,x\right]$$

$$= \frac{(\alpha+\beta-1)_\lambda}{(\beta)_\lambda}\; F^{p+1,q}_{r+1,s}\left[\begin{matrix} a_p,\ \alpha+\beta+\lambda-1 : b_q\,;\, b'_q\,; \\ c_r,\ \alpha+\beta-1 : d_s\,;\ d'_s\,; \end{matrix}\ \mu_1\,x,\ \mu_2\,x\right]. \qquad \ldots(2.1)$$

Proof of (2.1) — We know that

$$D_x^{\lambda}\left[x^{\alpha-1}\,x^{\beta+\lambda-1}\,F^{p,q}_{r,s}\left[\begin{matrix} a_p : b_q\,;\ b'_q\,; \\ c_r : d_s\ ;\ d'_s\,; \end{matrix}\ \mu_1\,x,\ \mu_2\,x\right]\right]$$

$$= \sum_{t=0}^{\infty}\binom{\lambda}{t}\,D_x^{t}\,(x^{\alpha-1})\,D_x^{\lambda-t}\left[x^{\beta+\lambda-1}\,F^{p,q}_{r,s}\left[\begin{matrix} a_p : b_q\,;\, b'_q\,; \\ c_r : d_s\,;\ d'_s\,; \end{matrix}\ \mu_1\,x,\ \mu_2\,x\right]\right].$$

$$\ldots(2.2)$$

Now, putting Kampé de Fériet functions of both sides in the form of double series with the help of (1.1) and then making use of result (1.3), one can get (2.1) after some simplifications.

3. Particular Cases

Putting $\mu_2 = 0$, $q = s$, $b_i = d_i$, $(i = 1, 2, \ldots, s)$ in (2.1) we get

$$\sum_{t=0}^{\lambda} \frac{(-\lambda)_t\,(1-\alpha)_t}{(1)_t\,(\beta)_t}\;{}_{p+1}F_{r+1}\,[(a_p),\ \beta+\lambda;\ (c_r),\ \beta+t;\ \mu_1\,x]$$

$$= \frac{(\alpha+\beta-1)_\lambda}{(\beta)_\lambda}\;{}_{p+1}F_{r+1}\,[(a_p),\ \alpha+\beta+\lambda-1;\ (c_r),\ \alpha+\beta-1;\ \mu_1\,x]. \qquad \ldots(3.1)$$

Putting $r = 1$, $c_1 = \beta + \lambda$, $p = 2$, $a_1 = a$, $a_2 = b$, $\mu_1 = x = 1$ in (3.1) and summing ${}_2F_1$ (1) series of left-hand side by known result due to (Slater 1966, Appendix III$_3$) we get :

$$\frac{\Gamma[\beta]\,\Gamma[\beta-a-b]}{\Gamma[\beta-a]\,\Gamma[\beta-b]}\;{}_3F_2\,[-\lambda,\ 1-\alpha,\ \beta-a-b;\ \beta-a,\ \beta-b;\ 1]$$

$$= \frac{(\alpha+\beta-1)_\lambda}{(\beta)_\lambda}\;{}_3F_2\,[a,\ b,\ \alpha+\beta+\lambda-1;\ \beta+\lambda,\ \alpha+\beta-1;\ 1]. \qquad \ldots(3.2)$$

This gives a transformation of ${}_3F_2\,(1)$ series into a similar series.

Putting $r=1$, $c_1=\alpha+\beta+\lambda-1$, $p=2$, $a_1=-n$, $a_2=b$ and $\mu_1=x=1$ in (3.1) and summing the right-hand side by making use of (Slater 1966, Appendix III_3) we get

$$\sum_{t=0}^{\lambda}\sum_{m=0}^{n}\frac{(-\lambda)_t\,(-n)_m\,(1-\alpha)_t\,(b)_m\,(\beta+\lambda)_m}{(1)_t\,(1)_m\,(\alpha+\beta+\lambda-1)_m\,(\beta)_{t+m}}=\frac{(\alpha+\beta-1)_\lambda\,(\alpha+\beta-b-1)_n}{(\beta)_\lambda\,(\alpha+\beta-1)_n}. \quad ...(3.3)$$

Similarly, putting $p=2$, $a_1=-n$, $a_2=\alpha+\beta-1$, $r=1$, $c_1=c$ and $\mu_1=x=1$ and summing the right-hand side by making use of (Slater 1966, Appendix III_3) we get

$$\sum_{t=0}^{\lambda}\sum_{m=0}^{n}\frac{(-\lambda)_t\,(-n)_m\,(1-\alpha)_t\,(\beta+\lambda)_m\,(\alpha+\beta-1)_m}{(1)_t\,(1)_m\,(\beta)_{t+m}\,(c)_m}$$

$$=\frac{(\alpha+\beta-1)_\lambda\,(c-\alpha-\beta-\lambda+1)_n}{(\beta)_\lambda\,(c)_n}. \quad ...(3.4)$$

Putting $p=r=0$, $q=2$, $b_1=-u$, $b_2=\alpha-1$, $b'_1=-v$, $b'_2=\beta$, $s=1$, $d_1=\alpha+\beta+\lambda-1$, $d'_1=\alpha+\beta+\lambda-1$ and $\mu_1=\mu_2=x=y=1$ in (2.1) and summing the right-hand side by making use of the result due to Cartlitz (1967) we get

$$\sum_{t=0}^{\lambda}\sum_{m=0}^{u}\sum_{n=0}^{v}\frac{(-\lambda)_t\,(-u)_m\,(-v)_n\,(\beta+\lambda)_{m+n}\,(1-\alpha)_t\,(\alpha-1)_m\,(\beta)_n}{(1)_t\,(1)_m\,(1)_n\,(\beta)_{m+n+t}(\alpha+\beta+\lambda-1)_m\,(\alpha+\beta+\lambda-1)_n}$$

$$=\frac{(\alpha+\beta-1)_{u+v+\lambda}\,(\beta)_u\,(\alpha-1)_v\,(\alpha+\beta-1)_\lambda\,(\alpha+\beta-1)_\lambda}{(\alpha+\beta-1)_{u+v}\,(\alpha+\beta-1)_{u+\lambda}\,(\alpha+\beta-1)_{v+\lambda}\,(\beta)_\lambda}. \quad ...(3.5)$$

Again, putting $p=r=1$, $a_1=\alpha+\beta-1$, $c_1=\alpha+\beta+\lambda-1$, $q=2$, $b_1=-u$, $b_2=a$, $b'_1=-v$, $b'_2=b$, $s=1$, $d_1=c$, $d'_1=d$ and $\mu_1=\mu_2=x=y=1$ in (2.1) and summing the right-hand side by Gauss's summation theorem (Slater 1966, Appendix III_3,) we get

$$\sum_{t=0}^{\lambda}\sum_{m=0}^{u}\sum_{n=0}^{v}\frac{(-\lambda)_t\,(-u)_m\,(-v)_n\,(1-\alpha)_t\,(\alpha+\beta-1)_{m+n}\,(\beta+\lambda)_{m+n}\,(a)_m\,(b)_n}{(1)_t\,(1)_m\,(1)_n\,(\beta)_{m+n+t}\,(\alpha+\beta+\lambda-1)_{m+n}\,(c)_m\,(d)_n}$$

$$=\frac{(\alpha+\beta-1)_\lambda\,(c-a)_u\,(d-b)_v}{(\beta)_\lambda\,(c)_u\,(d)_v}. \quad ...(3.6)$$

A number of similar other interesting results can also be scored.

ACKNOWLEDGEMENT

We are thankful to Dr R. Y. Denis, Department of Mathematics, University of Gorakhpur, Gorakhpur, for his valuable guidance in the preparation of the present paper.

REFERENCES

Carlitz, L. (1967). A summation theorem for double hypergeometric series. *Rendi. Cant. Del. Seminario. Mathematico Della Universita Di Padove*, 230–33.

Jain, R. N. (1966). Sum of a double hypergeometric series. *Mathematiche* (*Catania*), **21** 300–301.

Manocha, H. C., and Sharma, B. L. (1974). Fractional derivatives and summation. *J. Indian Math. Soc.*, **38**, 371–82.

Singal, R. P. (1970). An expansion of Kampé de Fériet hypergeometric function. *Math. Education*, **4**, 90–92.

Slater, L. J. (1966). Generalized Hypergeometric Functions. Cambridge University Press, Cambridge.

Srivastava, G. P. (1971). Some new transformation and reducible cases of Appell's double series and their generalizations. *Math. Student.*, **39**, 319–28.