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CERTAIN RESULTS ON A GENERAL SEQUENCE OF FUNCTIONS

G. N. CHANDWANI

Department of Mathematics, B. S. P. Higher Secondary School, Sector X, Bhilai Nagar (M.P.)

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Following Srivastava and Singhal (1971, 1972), Chandel and Agrawal (1977) defined a general sequence of functions and investigated some operational relationships connecting such functions on the analogy of the results given earlier by Srivastava and Panda (1975). The author records a few more results which he derives here along the lines detailed by Srivastava and Panda (1975); see also the concluding remark.

1. INTRODUCTION

Following Srivastava and Panda (1975), Chandel and Agrawal (1977) gave an account of operational relations for a sequence of functions defined by

$$T_{n,e,f,g}^{[\alpha,\beta,\gamma,k]}(x; a, b, c, d, p, r) = \frac{(ax+b)^{-\alpha}\,(cx+d)^{-\beta}\,x^{-\gamma}\,e^{px^r}}{n\,!} \times \Omega_x^n\left[(ax+b)^{\alpha+en}\,(cx+d)^{\beta+fn}\,x^{\gamma+gn}\,e^{-px^r}\right] \qquad \ldots(1.1)$$

where $a, b, c, d, e, f, g, g, p, r, \alpha, \beta, \gamma, k$ are arbitrary constants independent of n, and

$$\left.\begin{aligned} \Omega_x &\equiv x^k \frac{d}{dx} \\ \Omega_x^n &\equiv \Omega_x^{n-1}\,.\,\Omega_x\,. \end{aligned}\right\} \qquad \ldots(1.2)$$

The purpose of this note is to present a few additional results on this sequence of functions.

We first record here some known formulae for ready reference [Srivastava and Singhal (1971, p.76)] :

$$\Omega_x^n\{u\,.\,v\} = \sum_{j=0}^{n} \binom{n}{j} \Omega_x^{n-j}\{u\}\,.\,\Omega_x^{j}\{v\} \qquad \ldots(1.3)$$

where u and v are functions of x differentiable any number of times and

$$e^{t\Omega_x}\{f(x)\} = f\left(x\{1-(k-1)\,t\,x^{k-1}\}^{-\frac{1}{(k-1)}}\right),\; k \neq 1 \qquad \ldots(1.4)$$

$$\Omega_x^n\{w(x)\,X^n\,\phi(x)\} = w(x) \prod_{j=1}^{n}\left[\left(X\frac{w'(x)}{w(x)} + j\,x'\right)x^k + X\,\Omega_x\right]\phi(x) \qquad \ldots(1.5)$$

where $w(x)$ is any function of x, X is any polynomial in x and $\frac{w'(x)}{w(x)}$ is a linear function of x.

The relation connecting the operators $D \equiv d/dx$ and $\delta \equiv x\frac{d}{dx}$ is

$$x^n D^n = \delta(\delta-1)(\delta-2)\ldots(\delta-n+1) = \prod_{j=1}^{n}(\delta-j+1). \qquad \ldots(1.6)$$

Srivastava and Singhal (1972) have proved the following properties of the operator

$$\delta \equiv x\frac{d}{dx}$$

$$f(\delta)[\exp\{g(x)\}\, h(x)] = \exp\{g(x)\}.\, f(\delta+xg').\, h(x). \qquad \ldots(1.7)$$

For brevity, we will write

$$\Gamma_{n,e,f,g}^{[\alpha,\beta,\gamma,k]} \text{ for } \Gamma_{n,e,f,j}^{[\alpha,\beta,\gamma,k]}(x, a, b, c, d, p, r).$$

2. Operational Formulae

Starting from the definition (1.1) and applying (1.3) we have

$$\Omega_x^n\left[(ax+b)^{\alpha+en}(cx+d)^{\beta+fn}x^{\gamma+gn}e^{-px^r}\phi(x)\right]$$

$$= \sum_{m=0}^{n}\binom{n}{m}(n-m)!\,(ax+b)^{\alpha+em}(cx+d)^{\beta+fm}x^{\gamma+gm}e^{-px^r}$$

$$\times\, \Gamma_{n-m,e,f,g}^{[\alpha+em,\ \beta+fm,\ \gamma+gm,\ k]}\,\Omega_x^m\,\phi(x). \qquad \ldots(2.1)$$

Now $\Omega_x = x^k\frac{d}{dx} = \frac{d}{du}$ where $u = x^{1-k}/1-k$ so that

$$\Omega_x^n = \left(\frac{d}{du}\right)^n = u^{-n}u^n\left(\frac{d}{du}\right)^n$$

$$\Omega_x^n = u^{-n}u^n\left(\frac{d}{du}\right)^n = u^n\prod_{j=1}^{n}(\delta-j+1) \qquad \ldots(2.2)$$

where $\delta = u\frac{d}{du}$ [from 1 5].

Using (2.2) on the left-hand side of (2.1) and applying (1.7) we can prove that

$$\prod_{j=1}^{n}\left[\delta-j+1+\frac{(\alpha+en)\,ax}{(ax+b)\,(1-k)}+\frac{(\beta+fn)\,cx}{(cx+d)\,(1-k)}+\frac{\gamma+gn}{1-k}-\frac{pr\,x^r}{1-k}\right]\phi(x)$$

(*equation continued on p.* 1169)

$$= \left(\frac{x^{1-k}}{1-k}\right)^n \sum_{m=0}^{n} (ax+b)^{(m-n)e} (cx+d)^{(m-n)f} x^{(m-n)g}$$

$$\times \Gamma_{n-m,e,f,g}^{[\alpha+em,\ \beta+fm,\ \gamma+gm,\ k]} \quad \Omega_x^{m} \phi(x). \qquad \ldots(2.3)$$

If $\phi(x) = 1$, (2.3) will give us

$$\prod_{j=1}^{n} \left[\delta + \frac{(\alpha+en)\ ax}{(ax+b)(1-k)} + \frac{(\beta+fn)\ cx}{(cx+d)(1-k)} + \frac{\gamma+gn}{1-k} - \frac{pr\ x^r}{1-k}\right]$$

$$= \left(\frac{x^{1-k}}{1-k}\right)^n (ax+b)^{-ne} (cx+d)^{-nf} x^{-ng}\, \Gamma_{n,e,f,g}^{[\alpha,\beta,\gamma,k]}.$$

3. Generating Function

Using (1.1) and exponential expansion, we have

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \Gamma_{n+m,e,f,g}^{(\alpha-(m+n)e,\ \beta-f(m+n),\ \gamma-g(m+n),\ k)}\ t^n$$

$$= \frac{1}{m!} (ax+b)^{-\alpha+em} (cx+d)^{-\beta+fm} x^{-\gamma+gm} e^{px^r}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \left\{(ax+b)^e (cx+d)^f x^g\ t\right\}^n \Omega_x^{m+n}[(ax+b)^\alpha (cx+d)^\beta x^\gamma e^{-px^r}]$$

$$= \frac{(ax+b)^{-\alpha+em} (cx+d)^{-\beta+fm} x^{-\gamma+gm} e^{px^r}}{m!}$$

$$\times \exp\{(ax+b)^e (cx+d)^f x^g\ t\ \Omega_x\} \{\Omega_x^m (ax+b)^\alpha (cx+d)^\beta x^\gamma e^{-px^r}\}$$

$$= (ax+b)^{-\alpha+em} (cx+d)^{-\beta+fm} x^{-\gamma+gm} \exp\{p\,x^r\}$$

$$\times \exp\{(ax+b)^e (cx+d)^f x^g\ t\ \Omega_x\} \{(ax+b)^{\alpha-em} (cx+d)^{\beta-fm} x^{\gamma-gm} e^{-px^r}\}$$

$$\times \Gamma_{m,e,f,g}^{[\alpha-em,\ \beta-fm,\ \gamma-gm,\ k]}.$$

Again using (1.4) we get the required generating function

$$\sum_{n=0}^{\infty} \binom{n+m}{m} \Gamma_{n+m,e,f,g}^{(\alpha-e(n+m),\ \beta-f(m+n),\ \gamma-g(m+n),\ k)}\ t^n$$

$$\left(\frac{ax\,A+b}{ax+b}\right)^{\alpha-em} \left(\frac{cx\,A+d}{cx+d}\right)^{\beta-fm} A^{\gamma-gm}$$

$$\times \exp\left(-p\,x^r (A^r - 1)\right) \quad \Gamma_{m,e,f,g}^{(\alpha-em,\ \beta-fm,\ \gamma-gm,\ k)} (x\,A,\ a,\ b,\ c,\ d,\ p,\ r)$$

where m is an integer $\geqslant 0$

$$A = [1-(k-1)\,x^{k-1}\,(a\,x+b)^{e}\,(c\,x+d)^{f}\,x^{g}.\ t\,]^{-\frac{1}{(k-1)}}.$$

4. Bilinear Generating Functions

We append a bilinear generating function and express the result in the form of

Theorem — 1 Let $R^{q}_{m,n}(y)$ be a polynomial of degree $\left[\frac{n}{q}\right]$ in y defined by

$$R^{q}_{m,n}(y) = \sum_{s=0}^{[n/q]} \left\{\frac{\lambda_{m,s}}{K_{m+qs}}\right\} y^{s} \binom{n}{q\,s}. \qquad \ldots(4.1)$$

Then for every integer $m \geqslant 0$ and $q \geqslant 1$

$$\sum_{n=0}^{\infty} \Gamma^{(\alpha-en,\ \beta-fn,\ \gamma-gn,\ k)}_{n,e,f,g}\ R^{q}_{m,n}(y)\ t^{n} \qquad \ldots(4.2)$$

$$=\left[\frac{ax\,A+b}{ax+b}\right]^{\alpha}\left[\frac{cx\,A+d}{cx+d}\right]^{\beta} x^{g}\ F_{q,m}\left[\,xA,\ yt^{q}\left(\frac{ax+b}{ax\,A+b}\right)^{e}\left(\frac{cx+d}{cx\,A+d}\right)^{f} x^{-gq}\right]$$

where

$$F_{q,m}[x,t] = \sum_{n=0}^{\infty} \frac{\lambda_{m,n}}{K_{m+qn}}\ \Gamma^{[\alpha-enq,\ \beta-fqn,\ \gamma-gqn,\ k]}_{nq,e,f,g}. \qquad \ldots(4.3)$$

Proof : We have

$$\sum_{n=0}^{\infty} \Gamma^{(\alpha-en,\ \beta-fn,\ \gamma-gn,\ k)}_{n,e,f,g}\ R^{q}_{m,n}(y)\ t^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{S=0}^{[n/q]} \binom{n}{qS} \frac{\lambda_{m,s}}{K_{m+qs}}\ y^{s}\ \Gamma^{[(\alpha-en,\ \beta-fn,\ \gamma-gn,\ k)]}_{n,e,f,g}\, t^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{S=0}^{\infty} \binom{n+qS}{qS} \frac{\lambda_{m,s}}{K_{m+qs}}\ \Gamma^{[\alpha-e(n+qs),\ \beta-f(n+qs),\ \gamma-g(n+qs),k]}_{n+qs,e,f,g}\ t^{n+qs}$$

$$\sum_{S=0}^{\infty} \frac{\lambda_{m,s}}{K_{m+qs}}\,(y\,t^{qs})\left[\frac{ax\,A+b}{ax+b}\right]^{\alpha-eqs}\left[\frac{cx\,A+d}{cx+d}\right]^{\beta-fqs} A^{\gamma-gqs}$$

$$\times\ e^{\cdot px^{r}}\,(A^{s}-1)\ \Gamma^{[\alpha-eqs,\ \beta-fqs,\ \gamma\ gqs,k]}_{qs,e,f,g}\ (xA,\ b,\ c,\ d,\ \gamma,\ s)$$

$= R.\ H.\ S.$

The result given above can be further generalised in the form of the following theorem.

Theorem 2 — For a polynomial defined by (1.1)

If $F_q(x, y, z, t) = \sum_{j=0}^{\infty} (a_j)\, g_j(y)\, T_{qj, e, f, g}^{[\alpha-eqj,\ \beta-fqj,\ \gamma-gqj,\ k]}(x, a, b, c, d, p, r)\, z^j t^{qj}$ ...(4.4)

Then $\sum_{n=0}^{\infty} T_{n, e, f, g}^{[\alpha-en,\ \beta-fn,\ \gamma-gn,\ k]}(x, a, b, c, d, p, r)\ \sigma_n^q(y, z)\, t^n$

$$= \left(\frac{ax\,A+b}{ax+b}\right)^{\alpha}\left(\frac{cx\,A+d}{cx+d}\right)^{\beta} A^r e^{px^r}(A^s-1)$$

$$\times F_q\left[x\,A, y, Z\,t^q\left(\frac{ax+b}{ax\,A+b}\right)^e\left(\frac{cx+d}{cx\,A+d}\right)^f x^{-gq}\right] \quad ...(4.5)$$

where

$$\sigma_n^q(y, z) = \sum_{j=0}^{[n/q]} (a_j)\binom{n}{qj} g_j(y)\, z^j. \quad ...(4.6)$$

Proof is the same as that of Theorem 1.

5. Particular Cases

Particularly for $k = r = g = 0$ and $e = f = 1$, the above result reduces to result due to Srivastava and Singhal (1972).

Remark : In view of the generating function (3.1), Theorems 1 and 2 of this paper are derivable as obvious special cases of a general result given recently by Srivastava (1980, p. 224, Theorem 2) where several classes of bilateral generating functions have been presented rather systematically.

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