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INTEGRABILITY CONDITIONS IN CYLINDRICALLY SYMMETRIC NON-STATIC EINSTEIN-ROSEN SPACE-TIMES

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Much attention has been directed towards the solution of vacuum Einstein equations when the space-time is axially symmetric and static. (For a general survey, the reader is referred to Kinnersley 1974, Sato and Tomimatsu 1972). In a recent paper, the present authors used certain integrability conditions in axially symmetric static space-times to arrive at an infinite family of vacuum solutions parametrized by 2-dimensional Harmonic functions (Nagaraj and Prabhakara 1981). In this paper we take a similar approach to the cylindrically symmetric non-static (Einstein-Rosen) space-times. It is shown that a necessary and sufficient condition for $R_2^2 + R_3^3 = 0$ is that there exists a function ψ (r, t) satisfying the wave equation $\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} \right) \psi = 0$. The equations $R_2^2 + R_3^3 = 0$, $R_1^1 - R_4^4 = 0$ and $R_1^4 = 0$ are shown to be sufficient to characterize empty cylindrically symmetric non-static (Einstein-Rosen) space-times.

1. INTRODUCTION

Let (M, g) be a real 4-dimensional C^∞-Einstein Riemann manifold with the structure tensor g which is a symmetric tensor field of type (0,2). The local signature of g is $(+ + + -)$. The Riemann tensor and Ricci tensor have the components $R_{ABC}^{\ \ D}$ and $R_{BC} \overset{\text{def}}{=} R_{ABC}^{\ \ A}$ w.r.t a coordinate system (x^A). The scalar curvature is denoted by $R \overset{\text{def}}{=} R_{BC} g^{BC}$.

The most general form of cylindrically symmetric non-static Einstein-Rosen space-time is given by

$$ds^2 = e^{2\alpha} (dr^2 - dt^2) + e^{2\beta} d\phi^2 + e^{2\gamma} dz^2 \qquad \ldots(1.1)$$

where α, β, γ are functions of r and t only. The components g_{AB} are :

$$g_{11} = e^{2\alpha},\ g_{22} = e^{2\beta},\ g_{33} = e^{2\gamma},\ g_{44} = -e^{2\alpha}, \qquad \ldots(1.2)$$

rest of $g_{AB} = 0$. The components g^{AB} are

$$g^{11} = e^{-2\alpha},\ g^{22} = e^{-2\beta},\ g^{33} = e^{-2\gamma},\ g^{44} = -e^{-2\alpha}, \qquad \ldots(1.3)$$

rest of $g^{AB} = 0$.

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We take $x^1 = r$, $x^2 = \phi$, $x^3 = z$, $x^4 = t$.

The non-vanishing components of Ricci tensor are calculated :
(here a comma denotes partial derivative)

$$\left.\begin{aligned}
R_{11} &= \alpha_{,11} - \alpha_{,44} + \beta_{,11} + \gamma_{,11} + \beta_{,1}^2 + \gamma_{,1}^2 \\
&\quad - \alpha_{,1}(\beta_{,1} + \gamma_{,1}) - \alpha_{,4}(\beta_{,4} + \gamma_{,4}). \\
R_{22} &= e^{2(\beta-\alpha)}[\beta_{,11} - \beta_{,44} + \beta_{,1}^2 - \beta_{,4}^2 + \beta_{,1}\gamma_{,1} - \beta_{,4}\gamma_{,4}]. \\
R_{33} &= e^{2(\gamma-\alpha)}[\gamma_{,11} - \gamma_{,44} + \gamma_{,1}^2 - \gamma_{,4}^2 + \beta_{,1}\gamma_{,1} - \beta_{,4}\gamma_{,4}]. \\
R_{44} &= \alpha_{,44} - \alpha_{,11} + \beta_{,44} + \gamma_{,44} + \beta_{,4}^2 + \gamma_{,4}^2 \\
&\quad - \alpha_{,1}(\beta_{,1} + \gamma_{,1}) - \alpha_{,4}(\beta_{,4} + \gamma_{,4}). \\
R_{14} &= \beta_{,14} + \gamma_{,14} + \beta_{,1}\beta_{,4} + \gamma_{,1}\gamma_{,4} - \alpha_{,1}(\beta_{,4} + \gamma_{,4}) \\
&\quad - \alpha_{,4}(\beta_{,1} + \gamma_{,1}).
\end{aligned}\right\} \quad \ldots(1.4)$$

Let $\Box$ denote the wave operator $\left(\frac{\partial^2}{\partial x^{1^2}} - \frac{\partial^2}{\partial x^{4^2}} \right)$.

Here we state and prove the following proposition.

Proposition—A necessary and sufficient condition for $R_2^2 + R_3^3 = 0$ is : $e^{\beta+\gamma}$ is a wave function : $\Box\,(e^{\beta+\gamma}) = 0$.

PROOF : Using eqns. (1.4) we see that a necessary and sufficient condition for $R_2^2 + R_3^3 = 0$ is $e^{-2\alpha}\,[(\beta + \gamma)_{,11} - (\beta + \gamma)_{,44} + \beta_{,1}^2 + \gamma_{,1}^2$

$$- \beta_{,4}^2 - \gamma_{,4}^2 + 2\beta_{,1}\gamma_{,1} - 2\beta_{,4}\gamma_{,4}] = 0.$$

This latter equation is simplified to :

$$e^{-(2\alpha+\beta+\gamma)} \,.\, \Box\,(e^{\beta+\gamma}) = 0.$$

Hence $\Box\,(e^{\beta+\gamma}) = 0$.

That is $e^{\beta+\gamma}$ is a wave function, say $\psi\,(x^1, x^4)$. Q.E.D.

We take $\alpha + \gamma = \lambda$ and write the values of R_A^B with $e^{\beta+\gamma} = \psi$, using eqns. (1.4) :

$$\left.\begin{aligned}
R_1^1 &= e^{-2(\lambda-\gamma)}\Big[\Box(\lambda - \gamma) + 2\gamma_{,1}^2 + \frac{\psi_{,11}}{\psi} - \frac{2\gamma_{,1}\psi_{,1}}{\psi} \\
&\quad - \frac{(\lambda-\gamma)_{,1}\psi_{,1}}{\psi} - \frac{(\lambda-\gamma)_{,4}\psi_{,4}}{\psi}\Big] \\
R_2^2 &= -R_3^3\, e^{-2(\lambda-\gamma)}\Big[-\Box\gamma - \frac{1}{\psi}(\gamma_{,1}\psi_{,1} - \gamma_{,4}\psi_{,4})\Big] \\
R_4^4 &= e^{-2(\lambda-\gamma)}\Big[\Box(\lambda - \gamma) - 2\gamma_{,4}^2 - \frac{\psi_{,44}}{\psi} + \frac{2\gamma_{,4}\psi_{,4}}{\psi} \\
&\quad + \frac{(\lambda-\gamma)_{,1}\psi_{,1}}{\psi} + \frac{(\lambda-\gamma)_{,4}\psi_{,4}}{\psi}\Big] \\
R_1^4 &= e^{-2(\lambda-\gamma)}\Big[-\frac{\psi_{,14}}{\psi} - 2\gamma_{,1}\gamma_{,4} + \frac{1}{\psi}(\lambda_{,1}\psi_{,4} + \lambda_{,4}\psi_{,1})\Big].
\end{aligned}\right\} \quad \ldots(1.5)$$

2. Partial Differential Equations for λ and their Integrability Conditions

We write the following equations using eqns. (1.5) :

$$
\left.
\begin{aligned}
&\text{(a)}\quad R^1_1 + R^4_4 = 2e^{-2(\lambda-\gamma)}\Big[\gamma_{,1} - \gamma_{,4}^2 + \Box(\lambda-\gamma) - \frac{(\gamma_{,1}\psi_{,1} - \gamma_{,4}\psi_{,4})}{\psi}\Big]\\
&\text{(b)}\quad R^1_1 - R^4_4 = 2e^{-2(\lambda-\gamma)}\Big[\gamma_{,}^{1} + \gamma_{,4} + \frac{\psi_{,11}}{\psi} - \frac{(\lambda_{,1}\psi_{,1} + \lambda_{,4}\psi_{,4})}{\psi}\Big]\\
&\text{(c)}\quad R^1_1 + R^4_4 - 2R^2_2 = 2e^{-2(\lambda-\gamma)}\left[\Box\lambda + \gamma_{,1}^2 - \gamma_{,4}^2\right]\\
&\text{(d)}\quad R^4_1 = e^{-2(\lambda-\gamma)}\Big[-\frac{\psi_{,14}}{\psi} - 2\gamma_{,1}\gamma_{,4} + \frac{\lambda_{,1}\psi_{,4} + \lambda_{,4}\psi_{,1}}{\psi}\Big].
\end{aligned}
\right\}\qquad \ldots(2.1)
$$

We consider the equations $R^1_1 - R^4_4 = 0$ and $R^4_1 = 0$.
Solving these equations for $\lambda_{,1}$ and $\lambda_{,4}$ we get

$$
\left.
\begin{aligned}
\lambda_{,1} &= (\ln A^{1/2})_{,1} + \frac{\psi}{A}\left[\psi_{,1}(\gamma_{,1}^2 + \gamma_{,4}^2) - 2\psi_{,4}\gamma_{,1}\gamma_{,4}\right]\\
\lambda_{,4} &= (\ln A^{1/2})_{,4} - \frac{\psi}{A}\left[\psi_{,4}(\gamma_{,1}^2 + \gamma_{,4}^2) - 2\psi_{,1}\gamma_{,1}\gamma_{,4}\right]
\end{aligned}
\right\}\qquad \ldots(2.2)
$$

where $A = \psi_{,1}^2 - \psi_{,4}^2$ is assumed to be different from zero.

Applying integrability condition $\lambda_{,14} - \lambda_{,41} = 0$ to eqns. (2.2), we get :

$$(\gamma_{,1}\psi_{,4} - \gamma_{,4}\psi_{,1})(\psi\,\Box\,\gamma + \gamma_{,1}\psi_{,1} - \gamma_{,4}\psi_{,4}) = 0.$$

That is, from (2.1),

$$\psi\, e^{2(\lambda-\gamma)} R^3_3 (\gamma_{,1}\psi_{,4} - \gamma_{,4}\psi_{,1}) = 0.$$

Therefore the integrability conditions are : $\ldots(2.3)$

(a) $R^3_3 = 0$ or (b) $\gamma_{,1}\psi_{,4} - \gamma_{,4}\psi_{,1} = 0$.

Case (i)—We consider the case $R^3_3 = 0 = -R^2_2$ and $\gamma_{,1}\psi_{,4} - \gamma_{,4}\psi_{,1} \neq 0$.

We calculate $\Box\,\lambda$ by using eqns. (2.2) :

$$\Box\lambda = -\gamma_{,1}^2 + \gamma_{,4}^2. \qquad \ldots(2.4)$$

Hence from (2.1c) it follows that $R^1_1 + R^4_4 = 0$.

And since $R^1_1 - R^4_4 = 0$, we get $R^1_1 = 0$ and $R^4_4 = 0$.

Therefore $R^B_A = 0$, since R^4_1 is assumed to be zero. Thus when γ does not satisfy the integrability condition (2.3b), the space-time is empty if and only if λ and γ are determined by the differential equations

$$\Box \lambda = -\gamma_{,1}^2 + \gamma_{,4}^2 \; ; \; \Box \gamma = -\frac{1}{\psi}\left(\psi_{,1}\gamma_{,1} - \psi_{,4}\gamma_{,4}\right).$$

Case (ii)—Next we consider the case $R^3_3 \neq 0$ and (2.3b) holds.
Equation (2.3b) is valid iff $\gamma = \gamma(\psi)$.
Using eqns. (2.2) and $\gamma = \gamma(\psi)$ we get

$$\left.\begin{aligned} \lambda_{,1} &= (\ln A^{1/2})_{,1} + \psi\psi_{,1}\left(\frac{d\gamma}{d\psi}\right)^2 \\ \lambda_{,4} &= (\ln A^{1/2})_{,4} + \psi\psi_{,4}\left(\frac{d\gamma}{d\psi}\right)^2. \end{aligned}\right\} \qquad \ldots(2.5)$$

Using eqns. (2.5) $\Box \lambda$ is calculated :

$$\Box \lambda = A \frac{d\gamma}{d\psi}\left(\frac{d\gamma}{d\psi} + 2\psi \frac{d^2\gamma}{d\psi^2}\right). \qquad \ldots(2.6)$$

From (2.6) and (2.1a), we have :

$$R^1_1 + R^4_4 = \frac{2A}{\psi} e^{-2(\lambda-\gamma)}\left(\frac{d\gamma}{d\psi} + \psi\frac{d^2\gamma}{d\psi^2}\right)\left(2\psi\frac{d\gamma}{d\psi} - 1\right).$$

Since $R^1_1 - R^4_4 = 0$ we have from the equation above,

$$R^1_1 = R^4_4 = \frac{A}{\psi} e^{-2(\lambda-\gamma)}\left(\frac{d\gamma}{d\psi} + \psi\frac{d^2\gamma}{d\psi^2}\right)\left(2\psi\frac{d\gamma}{d\psi} - 1\right). \qquad \ldots(2.7)$$

From (1.5), we have

$$R^2_2 = -R^3_3 = -2A\, e^{-2(\lambda-\gamma)}\left(\frac{d\gamma}{d\psi} + \psi\frac{d^2\gamma}{d\psi^2}\right). \qquad \ldots(2.8)$$

Note that

$$\frac{d\gamma}{d\psi} + \psi\frac{d^2\gamma}{d\psi^2} = \frac{1}{2}\frac{d}{d\psi}\left(2\psi\frac{d\gamma}{d\psi} - 1\right).$$

Therefore eqns. (2.7) and (2.8) together imply that $R^1_1 = R^4_4 = 0$ if and only if $R^2_2 = -R^3_3 = 0$. Since the assumption is that $R^2_2 \neq 0$, we have $R^1_1 = R^4_4 \neq 0$.

The most general solution of $\frac{d\gamma}{d\psi} + \psi\frac{d^2\gamma}{d\psi^2} = 0$ is $\gamma = \ln(c_2\psi^{c_1})$

where c_1 and c_2 are arbitrary constants of integration. It should be noted that this family of solutions includes the family of solutions $\gamma = \ln(c_2\psi^{1/2})$ of

$$2\psi\frac{d\gamma}{d\psi} - 1 = 0.$$

Thus when $R^4_1 = 0$, $R^1_1 = R^4_4 \neq 0$ and $R^2_2 = - R^3_3 \neq 0$, the integrability condition $\gamma = \gamma(\psi)$ is satisfied.

Then λ is given as a function of ψ alone :

$$\lambda = \ln A^{1/2} \left(\int \psi \left(\frac{d\gamma}{d\psi} \right)^2 d\psi \right)$$

for every wave function ψ ($\Box \psi = 0$) such that

$$A = \psi_{,1}^2 - \psi_{,4}^2 \neq 0.$$

Case (iii)—Lastly we consider the case when both the integrability conditions $R^3_3 = 0$ and $\gamma_{,1} \psi_{,4} - \gamma_{,4} \psi_{,1} = 0$ hold.

When $R^3_3 = 0$, we have shown in case (i) that $R^B_A = 0$.

Also by case (ii)

$$\gamma_{,1} \psi_{,4} - \gamma_{,4} \psi_{,1} = 0 \text{ gives}$$

$$\gamma = \ln (c_2 \psi^{c_1}) \text{ if and only if } R^B_A = 0. \qquad \text{...(2.9)}$$

Hence the space-time is empty if

$$\gamma = \ln (c_2 \psi^{c_1}).$$

Now we shall give an explicit exact solution for the cylindrically symmetric non-static (Einstein-Rosen) vacuum space-time in terms of the wave function ψ. Substituting the value of γ given by (2.9) in eqn. (2.2), λ is calculated :

$$\lambda = \ln (c_3 A^{1/2} \psi^{c_1^2}) \qquad \text{...(2.10)}$$

where c_3 is a constant of integration.

Therefore the line element (1.1) takes the form

$$ds^2 = \left(\frac{c^3}{c_2} \right)^2 A \, \psi^{2c_1(c_1-1)} \left({dx^1}^2 - {dx^4}^2 \right) + \frac{1}{c_3^2} \psi^{2(1-c_1)} {dx^2}^2 + c_2^2 \psi^{2c_1} {dx^3}^2 . \qquad \text{...(2.11)}$$

Thus in (2.11) we have an infinite family of vacuum cylindrically symmetric non-static (Einstein-Rosen) space-times each corresponding to the wave function ψ.

3. Case when $A \equiv \psi_{,1}^2 - \psi_{,4}^2 = 0$

In section 2, we considered the integrability condition for the partial differential equations for λ when $A = \psi_{,1}^2 - \psi_{,4}^2 \neq 0$. In this section our aim is to discuss the situation that arises when $A \equiv \psi_{,1}^2 - \psi_{,4}^2 = 0$. In this case we get $R^2_2 = 0$ always. Further, a vacuum solution is always defined by taking λ, γ as functions of either the retarded time or the advanced time.

The null coordinates ξ and η defined in the following, prove to be useful :

$$\xi = \frac{x^1 + x^4}{\sqrt{2}}, \quad \eta = \frac{x^1 - x^4}{\sqrt{2}}. \qquad \text{...(3.1a)}$$

The coordinates x^1 and x^4 are then given by

$$x^1 = \frac{\xi + \eta}{\sqrt{2}} \text{ and } x^4 = \frac{\xi - \eta}{\sqrt{2}} . \qquad \text{...(3.1b)}$$

The metric (1.1) becomes

$$2e^{2(\lambda-\gamma)}\, d\xi\, d\eta + \psi^2 e^{-2\gamma}\, dx^{2^2} + e^{2\gamma}\, dx^{3^2} \qquad \text{...(3.2)}$$

where $e^{\beta+\gamma} = \psi$ in (1.1) and ψ is again a wave function satisfying $\Box\, \psi = 0$. Here we shall assume that $A = \psi_{,1}^2 - \psi_{,4}^2 \equiv 0$. In terms of the null coordinates (ξ, η) $\Box\, \psi = 0$ now becomes

$$\psi_{,\xi\eta} = 0. \qquad \text{...(3.3a)}$$

The most general solution of (3.3a) is given by

$$\psi\, (\xi, \eta) = g(\xi) + h\, (\eta), \qquad \text{...(3.3b)}$$

where $g(\xi)$ and $h(\eta)$ are arbitrary smooth functions.
Then $A \equiv \psi_{,1}^2 - \psi_{,4}^2 = 0$ gives

$$g_{,\xi}\, h_{,\eta} = 0. \qquad \text{...(3.3c)}$$

Thus when $A = 0$ we get two distinct cases
(i) $g(\xi) \equiv 0$ and (ii) $h(\eta) \equiv 0$.

Using (1.5) and (3.1b) R_A^B are calculated :

$$\left.\begin{aligned}
R_1^1 &= e^{-2(\lambda-\gamma)}\Big[\, 2\,(\lambda_{,\xi\eta} - \gamma_{,\xi\eta}) + (\gamma_{,\xi} + \gamma_{,\eta})^2 \\
&\quad - \frac{(\gamma_{,\xi}\, \psi_{,\eta} + \gamma_{,\eta}\, \psi_{,\xi})}{\psi} + \frac{1}{2\psi}\,(\psi_{,\xi\xi} + 2\psi_{,\xi\eta} \\
&\quad + \psi_{,\eta\eta}) - \frac{(\lambda_{,\xi}\, \psi_{,\xi} + \lambda_{,\eta}\, \psi_{,\eta})}{\psi}\Big]. \\
R_2^2 &= -R_3^3 = e^{-2(\lambda-\gamma)}\Big[-2\gamma_{,\xi\eta} - \frac{1}{\psi}\,(\gamma_{,\xi}\, \psi_{,\eta} + \gamma_{,\eta}\, \psi_{,\xi})\Big] \\
R_4^4 &= e^{-2(\lambda-\gamma)}\, [2\,(\lambda_{,\xi\eta} - \gamma_{,\xi\eta}) - (\gamma_{,\xi} - \gamma_{,\eta})^2 \\
&\quad - \frac{(\gamma_{,\xi}\, \psi_{,\eta} + \gamma_{,\eta}\, \psi_{,\xi})}{\psi} - \frac{1}{2\psi}\,(\psi_{,\xi\xi} - 2\psi_{,\xi\eta} + \psi_{,\eta\eta}) \\
&\quad + \frac{(\lambda_{,\xi}\, \psi_{,\xi} + \lambda_{,\eta}\, \psi_{,\eta})}{\psi}\Big] \\
R_1^4 &= e^{-2(\lambda-\gamma)}\Big[\frac{-1}{2\psi}\,(\psi_{,\xi\xi} - \psi_{,\eta\eta}) - (\gamma_{,\xi}^2 - \gamma_{,\eta}^2) \\
&\quad + \frac{1}{\psi}\,(\lambda_{,\xi}\, \psi_{,\xi} - \lambda_{,\eta}\, \psi_{,\eta})\Big].
\end{aligned}\right\} \qquad \text{...(3.4)}$$

Using eqns. (3.4) the following equations are written

$$\left.\begin{aligned} R_1^1 - R_4^4 - 2R_1^4 &= 4e^{-2(\lambda-\gamma)}\left[\frac{\psi_{,\xi\xi}}{2\psi} + \gamma_{,\xi}^2 - \frac{1}{\psi}\lambda_{,\xi}\psi_{,\xi}\right] \\ R_1^1 - R_4^4 + 2R_1^4 &= 4e^{-2(\lambda-\gamma)}\left[\frac{\psi_{,\eta\eta}}{2\psi} + \gamma_{,\eta}^2 - \frac{1}{\psi}\lambda_{,\eta}\psi_{,\eta}\right]. \end{aligned}\right\} \quad \ldots(3.5)$$

The equations $R_1^1 - R_4^4 = 0$ and $R_1^4 = 0$ imply from eqns. (3.5) that

$$\left.\begin{aligned} &\text{(a)} \quad \lambda_{,\xi}\psi_{,\xi} = \frac{1}{2}(\psi_{,\xi\xi} + 2\psi\gamma_{,\xi}^2) \\ &\text{(b)} \quad \lambda_{,\eta}\psi_{,\eta} = \frac{1}{2}(\psi_{,\eta\eta} + 2\psi\gamma_{,\eta}^2). \end{aligned}\right\} \quad \ldots(3.6)$$

Case (i) : When $g(\xi) = 0$, $\psi = h(\eta)$—Therefore from eqn. (3.6a), we get $\gamma_{,\xi}^2 = 0$ which implies that $\gamma = \gamma(\eta)$.

In this case λ is given by (using (3.6b))

$$\lambda = \ln(\psi_{,\eta})^{1/2} + \int \frac{\psi\gamma_{,\eta}^2}{\psi_{,\eta}} d\eta + m(\xi) \quad \ldots(3.7)$$

where $m(\xi)$ is an arbitrary function of ξ.

Case (ii) : When $h(\eta) = 0$, $\psi = g(\xi)$—Therefore from eqn. (3.6b) we get $\gamma_{,\eta}^2 = 0$ which implies that $\gamma = \gamma(\xi)$.

In this case λ is given by [using (3.6a)]

$$\lambda = \ln(\psi_{,\xi})^{1/2} + \int \frac{\psi\gamma_{,\xi}^2}{\psi_{,\xi}} d\xi + n(\eta) \quad \ldots(3.8)$$

where $n(\eta)$ is an arbitrary function of η.

We shall now give the expression for ds when λ and γ are functions of the retarded time and the advanced time. When the metric coefficients λ and γ are functions of the retarded time, the line element (3.2) takes the form :

$$ds^2 = 2\psi_{,\xi}\exp\left[2\left(\int \frac{\psi\gamma_{,\xi}^2}{\psi_{,\xi}} d\xi\right) - 2\gamma(\xi) + 2n(\eta)\, d\xi d\eta\right] + \psi^2 e^{-2\gamma} dx^{2^2} + e^{2\gamma} dx^{3^2}. \quad \ldots(3.9)$$

The line element (3.9) can be now written as

$$ds^2 = d\mu\, d\delta + \psi^2 e^{-2\gamma(\xi)} dx^{2^2} + e^{2\gamma(\xi)} dx^{3^2} \text{ where}$$

$$d\mu = \psi_{,\xi}\exp\left[2\int \frac{\psi\gamma_{,\xi}^2}{\psi_{,\xi}} d\xi - 2\gamma(\xi)\right] d\xi \text{ and} \quad \ldots(3.10)$$

$$d\delta = 2e^{2n(\eta)} d\eta.$$

Similarly when the metric coefficients λ and γ are functions of the advanced time, the line element (3.2) takes form :

$$ds^2 = 2\psi_{,\eta}\exp\left[2\left(\int \frac{\psi_{,}\gamma_{\eta}^2}{\psi_{,\eta}} d\eta\right) - 2\gamma(\eta) + 2m(\xi)\right] d\xi\, d\eta. + \psi^2 e^{-2\gamma} dx^{2^2} + e^{2\gamma} dx^{3^2}. \quad \ldots(3.11)$$

This can be written as

$$ds^2 = d\mu \, d\delta + \psi^2 \, e^{-2\gamma} \, {dx^2}^2 + e^{2\gamma} \, {dx^3}^2 \text{ where} \quad \ldots(3.12)$$

$$d\mu = \psi_{,\eta} \exp\left[2 \int \frac{\psi\gamma_{,\eta}}{\psi_{,\eta}} \, d\eta - 2\gamma(\eta) \right] d\eta \text{ and}$$

$$l\delta = 2e^{2m(\xi)} \, d\xi.$$

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