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CURVATURE COLLINEATIONS IN CERTAIN COSMOLOGICAL SPACE-TIMES

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It has been shown that the cosmological universes of Narlikar and Karmarkar, and Kasner admit proper curvature collineations. The curvature collineation vectors have been determined explicitly.

1. INTRODUCTION

It is generally recognized that the knowledge of conservation laws is of fundamental importance in the physical description of nature. It has been observed that the existence of certain geometric symmetries definable in terms of Lie derivatives lead to conservation laws. The curvature collineation symmetry (CC) introduced by Katzin *et al.* (1969) is defined by the condition that the V_n admits a vector ξ^i for the infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta t \qquad \ldots(1.1)$$

such that

$$\mathcal{L}_\xi R^h_{ijk} = 0, \qquad \ldots(1.2)$$

δt being infinitesimal. An investigation of this symmetry was strongly motivated by the all-important role of the Riemann curvature tensor in the theory of general relativity. The CC, in general relativity, signifies not only a type of geometrical symmetry of the space-time, but it also implies that the gravitational properties of the field are preserved along the CC vector.

Equation (1.2) can also be expressed in terms of partial derivatives as

$$\mathcal{L}_\xi R^h_{ijk} \equiv R^h_{ijk,m}\,\xi^m - R^m_{ijk}\,\xi^h_{,m} + R^h_{mjk}\,\xi^m_{,i} + R^h_{imk}\,\xi^m_{,j}$$
$$+ R^h_{ijm}\,\xi^m_{,k} = 0 \qquad \ldots(1.3)$$

where $\mathcal{L}_\xi$ stands for the Lie derivative with respect to the field vector ξ^i. A comma denotes ordinary denotes ordinary differentiation.

Besides the CC symmetry several other geometrical symmetries such as Motion, Homothetic Motion Conformal Motion. Affine collineation etc. have been discussed by Katzin *et al.* (1969). We call a CC vector proper if it does admit higher symmetries such as those mentioned above. Singh and Shri Ram (1975) have found that the Godel universe and the steady-state cosmological universe of Bondi and Gold admit Adler *et al.* (1965) proper curvature collineations.

In this paper the cosmological models of Narlikar and Karmarkar (1946) and Kasner (1921) have been examined for CC and it has been found that these universes admit proper curvature collineations.

2. Narlikar and Karmarkar Space-time

Narlikar and Karmarkar (1946), in search of a curious solution of Einstein's field equations, obtained the metric

$$ds^2 = dt^2 - (1 + kt)^p\, dx^2 - (1 + kt)^q\, dy^2 - (1 + kt)^r\, dz^2 \qquad ...(2.1)$$

where k is a constant, and the constants p, q, r are connected by the relations

$$p + q + r = 2, \quad pq + qr + rp = 0. \qquad ...(2.2)$$

Here x, y, z, t correspond to the coordinates x^1, x^2, x^3, x^4 respectively. The asymmetric field given by the metric (2.1) is being regarded as cosmological model or as a transitional model for a finite portion of space, that is something like a vacuum picket into which matter is rushing from the surrounding portions of the extra-galactic nebulae.

For the metric (2.1) the non-vanishing components of the Christofel symbols Γ^i_{jk} are

$$\Gamma^1_{14} = \frac{1}{2} kp\,(1 + kt)^{-1},\ \Gamma^2_{24} = \frac{1}{2} kq\,(1 + kt)^{-1},\ \Gamma^3_{34} = \frac{1}{2} kr\,(1 + kt)^{-1}$$

$$\Gamma^4_{11} = \frac{1}{2} kp\,(1 + kt)^{p-1},\ \Gamma^4_{22} = \frac{1}{2} kq\,(1 + kt)^{q-1}$$

$$\Gamma^4_{33} = \frac{1}{2} kr\,(1 + kt)^{r-1} \qquad ...(2.3)$$

The only non-vanishing components of the mixed curvature tensor for (2.1) are

$$R^1_{212} = \frac{1}{4} k^2 pq\,(1 + kt)^{q-2},\ R^1_{313} = \frac{1}{4} k^2 pr\,(1 + kt)^{r-2}$$

$$R^1_{414} = \frac{1}{2} k^2 p\,(1 - p/2)\,(1 + kt)^{-2},\ R^2_{121} = \frac{1}{4} k^2 pq\,(1 + kt)^{p-2}$$

$$R^2_{323} = \frac{1}{4} k^2 qr\,(1 + kt)^{r-2},\ R^2_{424} = \frac{1}{2} k^2 q\,(1 - q/2)\,(1 + kt)^{-2}$$

$$R^3_{434} = \frac{1}{2} k^2 r\,(1 - r/2)\,(1 + kt)^{-2},\ R^4_{141} = \frac{1}{2} k\,p\,(p/2 - 1)\,(1 + kt)^{p-2}$$

$$R^4_{242} = \frac{1}{2} k^2 q\,(q/2 - 1)\,(1 + kt)^{q-2}\ ,\ R^4_{343} = \frac{1}{2} k^2 r\,(r/2 - 1)\,(1+kt)^{r-2}$$

$$R^3_{131} = \frac{1}{4} k^2 pr\,(1 + kt)^{p-2}\ ,\ R^3_{232} = \frac{1}{4} k^2 qr\,(1 + kt)^{q-2}. \qquad \ldots(2.4)$$

From the algebraic symmetries on the indices in a V_4 we find that (1.1) represent 96 equations. Considering these equations for (2.1) we obtain the following set of equations.

The vanishing components of the Riemann curvature tensor, after some simplification, give the following set of equations :

$$\left.\begin{array}{l} \xi^1_{,2} = \xi^1_{,3} = \xi^1_{,4} = 0, \quad \xi^2_{,1} = \xi^2_{,3} = \xi^2_{,4} = 0, \\ \xi^3_{,1} = \xi^3_{,2} = \xi^3_{,4} = 0, \quad \xi^4_{,1} = \xi^4_{,2} = \xi^4_{,3} = 0. \end{array}\right\} \qquad \ldots(2.5)$$

From these equations we conclude that ξ^1, ξ^2 ξ^3 and ξ^4 are functions of x. y, z and t respectively.

For the non-vanishing components of R^h_{ijk} (1.1) give on simplification, the following four independent equations :

$$\xi^4_{,4} = k\ (1 + kt)^{-1}\,\xi^4\ , \qquad \ldots(2.6)$$

$$\xi^1_{,1} = \frac{1}{2} k\,(2 - p)\,(1 + kt)^{-1}\,\xi^4, \qquad \ldots(2.7)$$

$$\xi^2_{,2} = \frac{1}{2} k\,(2 - q)\,(1 + kt)^{-1}\,\xi^4, \qquad \ldots(2.8)$$

$$\xi^3_{,3} = \frac{1}{2} k\,(2 - r)\,(1 + kt)^{-1}\,\xi^4. \qquad \ldots(2.9)$$

Redundant and trivial equations have been omitted.

Equation (2.6) give the solution for ξ^4 as

$$\xi^4 = \alpha\,(1 + kt) \qquad \ldots(2.10)$$

where α is an integration constant. In view of (2.10) eqns. (2.7), (2.8) and (2.9) give the solutions for ξ^1, ξ^2 and ξ^3 as

$$\xi^1 = \frac{1}{2}\alpha\,(2 - p)\,kx + \beta,$$

$$\xi^2 = \frac{1}{2}\alpha\,(2 - q)\,ky + \gamma,$$

$$\xi^3 = \frac{1}{2}\alpha\,(2 - \gamma)\,\kappa z + \delta, \qquad \ldots(2.11)$$

where β, γ and δ are arbitrary constants of integration.

Hence, the space-time (2.1) admits the CC vector given by (2.10) and (2.11).

In order to decide whether this CC vector is proper we define

$$h_{ij} \equiv \mathcal{L}_\xi\, g_{ij} = \xi_{i;j} + \xi_{j;i} \qquad \ldots(2.12)$$

where a semicolon denotes covariant differentiation.
Then by the use of (2.10) and (2.11) we see that

$$h_{11} = 2\alpha k\,(1 + kt)^{p} \neq 0,$$

which shows that the CC vector ξ^i does not define motion.
For affine collineation the vector ξ must satisfy $h_{ij;k} = 0$ in general. Here we find that

$$h_{11;1} = -\,k^2 p^2\,[\alpha\,(2-p)\,kx + \beta]\,(1 + kt)^{2(p-1)}$$

which implies that the CC vector ξ^i is not an affine collineation vector.

It can easily be seen that $\xi^1_{;114} \neq 0$ which shows that ξ^i does not admit either a conformal motion or projective and conformal collineation. Hence the CC vector given by (2.10) and (2.11) is proper.

3. Kasner Metric

Proceeding as in section 2, we obtain, for the Kasner's empty, nonflat and anisotropic universe (Kasner 1921):

$$ds^2 = dt^2 - t^{4/3}\,(dx^2 + dy^2) - t^{-2/3}\,dz^2 \qquad \ldots(3.1)$$

the proper CC vector ξ^i given by

$$\left.\begin{aligned} \xi^1 &= \frac{1}{3}\,ax + b\,, \quad \xi^2 = \frac{1}{3}\,ay + c\,, \\ \xi^3 &= \frac{4}{3}\,az + d\,, \quad \xi^4 = at \end{aligned}\right\} \qquad \ldots(3.2)$$

where a, b, c and d are constants of integration.

It may be noted that for $p = q = 4/3$, $q = -2/3$ this vector reduces to the CC vector for the universe (2.1)

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