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A CLASS OF SOLUTIONS OF VACUUM FIELD EQUATIONS

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Solutions of vacuum field equations which admit a time like hypersurface orthogonal Killing vector ξ^a with it's norm ξ having the property $g^{ij}\,\xi_{,i}\,\xi_{,j} = F^2(\xi)$ are found. Unlike other methods a frame of three space like vectors is used to convert the second order partial differential equations of vacuum field equations into a set of two first order partial differential equations which are solved one after the other to obtain all the solutions with the above mentioned property. It is found that Schwarschild spherical mass and Weyl's line mass are the only solutions of vacuum field equations which have the above mentioned property.

Let ξ^a be a time like hypersurface orthogonal Killing vector. By choosing coordinates adopted to ξ^a we could write the fundamental metric tensor as (Adler 1965)

$$ds^2 = -\xi^2 dt^2 + g_{ij}\, dx^i\, dx^j;\quad i, j = 1, 2, 3 \qquad \ldots(1)$$

where norm ξ and g_{ij} are functions of the three space variables only and $\xi^a = \delta^a_t$. Because of the usual signature of the space time we assume that $g_{ij}dx^i dx^j$ is positive definite. We can regard the space time as built up of identical layers of three dimensional space hypersurfaces, all being orthogonal to time like coordinate curves. Because of the time independence of g_{ij} all three dimensional spaces are isometric with the three dimensional metric tensor g_{ij} and by avoiding the unessential time coordinate we could identify them with a three dimensional base space with a metric tensor g_{ij} and a scalar field $\xi(x^i)$. Einstein's field equations in vacuum for the space described by (1) are given by (Adler 1965, Perjes 1970, Kota and Perjes 1972)

$$R_{it} = 0 \qquad \ldots(2)$$

$$R_{tt} = g^{ij}\, p_{i;j} + F_{,i}\, p^i = 0 \qquad \ldots(3)$$

$$R_{ij} = R^*_{ij} + \frac{F p_{i;j}}{\xi} + \frac{F_{,j}\, p_i}{\xi}\,. \qquad \ldots(4)$$

where R^*_{ij} is the Ricci tensor of the three space, (;) represents the covariant derivative in the three space and we have introduced a vector p_i in the three space by

$$p_i = \xi_{,i}/F(\xi). \qquad \ldots(5)$$

As we are interested only in the solutions for which $g^{ij}\,\xi_{,i}\xi_{,j} = F^2(\xi)$ it is straight forward to show that p^i must be parallely transported (i.e. $p^i_{;j}\,p^j = 0$). Let us also introduce a complex vector u^i in the three space, parallely transported along p^i such that it satisfies the following orthonormality relations

$$\left.\begin{aligned} &u^i u_i = u^i p_i = 0,\ u^i \bar{u}_i = p^i p_i = 1, \\ &u^i = (q^i + in^i)/\sqrt{2} \end{aligned}\right\} \qquad \ldots(6)$$

where q^i and n^i are unit space like orthogonal vectors, and $\bar{u}^i$ is complex conjugate of u^i. Metric tensor g^{ij} of three space could be expressed in terms of u^i and p^i by

$$g^{ij} = p^i p^j + u^i \bar{u}^j + \bar{u}^i u^j$$

$$g_{ij} = p_i p_j + \bar{u}_i u_j + u_i \bar{u}_j\,. \qquad \ldots(7)$$

Field equations (2), (3) and (4) when expressed in terms of the frame of u^i and p^i become

$$R_{tt} = 2F\theta + F' = 0\,. \qquad \ldots(8)$$

Six equations in equation (4) become

$$\sigma' + 2\sigma\theta + \frac{F\sigma}{\xi} = 0 \qquad \ldots(9)$$

$$\theta' + \theta^2 + \sigma\bar{\sigma} + (F'/\xi) = 0 \qquad \ldots(10)$$

$$\theta_{,i}\bar{u}^i - \sigma_{,i}u^i + 2\sigma\bar{\zeta} = 0 \qquad \ldots(11)$$

$$\bar{\zeta}_{,i}u^i + \zeta_{,i}\bar{u}^i + 2\zeta\bar{\zeta} - \theta' - 2\theta^2 - F\theta/\xi = 0 \qquad \ldots(12)$$

where (′) denotes differentiation with respect to the parameter of the curve to which p^i is tangent and we have put

$$\theta = \bar{u}^i p_{i;j} u^j \qquad \ldots(13)$$

$$\sigma = \bar{u}^i p_{i;j} \bar{u}^j \qquad \ldots(14)$$

$$\zeta = u^i \bar{u}_{i;j} u^j\,. \qquad \ldots(15)$$

It is important to note at this point from (5) and (6)

$$\xi' = F(\xi),\ \xi_{,i}u^i = 0\,. \qquad \ldots(16)$$

Procedure for solving these two sets of first order differential equations (8) to (12) and (13) to (15) is as follows. Using eqn. (16) one integrates eqns. (8), (9) and (10) to obtain θ, σ and F. Equation 11 then gives the value of ζ. Knowing θ, σ, ζ and F one can then solve eqns. (13), (14) and (15) and integrate u^i along the curve to which p^i is tangent. Constants of integration are then related by the remaining eqn. (12) So far we have not restricted ourself to a particular type of coordinates in the three space. We could however, choose a coordinate system such that

$$p^i = \delta^i_1\,. \qquad \ldots(17)$$

Metric tensor g_{ij} could easily be calculated knowing u^i and p^i. One can integrate (16) to obtain ξ.

Let us now calculate u^i, F and ξ. Using (5), (16) and (17) we get from eqns. (8), (9) and (10) after integration

$$\theta = -F'/2F \quad ...(18)$$

$$\sigma = cF/\xi \quad ...(19)$$

$$F^{1/2} = A\ \dot{\xi}^m + B\ \xi^z \quad ...(20)$$

where (.) represents the differentiation w.r.t. x^1, A, B and c are constants of integration, m and l are given by

$$m = 1 + (c^2+1)^{1/2},\ z = 1 + (c^2+1)^{1/2}. \quad ...(21)$$

Using eqns. (16), (18), (19), (20) and eqn. (9) we get from eqn. (11)

$$2\sigma\zeta = 0 \quad ...(22)$$

which implies that either

$$\sigma = 0 \quad ...(23)$$

or $\quad \zeta = 0. \quad ...(24)$

At this point we divide the problem in two cases corresponding to eqns (23) and (24)

Case I : $\sigma = 0$.

Using (18), (23), (6) and (7) we can now solve eqns. (13) and (14), we get after some simplification

$$u_i\dot{u}^i = p_i\dot{u}^i = 0 \quad (25)$$

$$\bar{u}_i\dot{u}^i = \dot{F}/2F. \quad ..(26)$$

With the help of eqns. (6), (7) we get

$$\dot{u}^i\ u^i\ \dot{F}/2F \quad ...(27)$$

which on integration yields

$$u^i = F^{1/2}\ z^i\ (x^2, x^3) \quad ...(28)$$

where $z(x^2, x^3)$ are functions of x^2 and x^3 only. Since p_i is an hypersurface orthogonal vector we can by the right choice of coordinates in three space make

$$u^2 = F^{1/2}\ \underline{R}\ (x^2, x^3),\ u^3 = iF^{1/2}\ I\ (x^2, x^3),\ u^1 = 0, \quad ...(29)$$

where $R\ (x^2, x^3)$ and $I\ (x^2, x^3)$ are real functions of x^2 and x^3 only. By requiring that the remaining eqn. (12) be satisfied we get after using eqn. (15)

$$R\ (RI_{,2}/I)_{,2} + I\ (IR_{,3}/R)_{,3} - (RI_{,2}/I)^2 - (IR_{,3}/R)^2 + 4AB = 0. \quad ...(30)$$

Metric tensor of the space time could easily with the help of eqns. (5), (6), (7), (17) and (29) be written as

$$ds^2 = -\xi^2 dt^2 + (dx^1)^2 + (dx^2)^2/FR^2 + (dx^3)^2/FI^2 \quad ...(31)$$

where ξ and F' are given by equations (16) and (20). By rescaling the coordinates and making a coordinate transformation $x^1 = f(\underline{x}^1)$ such that $\xi(\underline{x}^1) = (1 - (2M/\underline{x}^1)^{1/2}$ plus the right choice of coordinates x^2 and x^3 we can put (31) in a following form

$$ds^2 = -(1 - 2M/\underline{x}^1)dt^2 + (1 - 2M/\underline{x}^1)^{-1}(d\underline{x}^1)^2 + (\underline{x}^1)^2\{(d\underline{x}^2)^2 + (\sin x^2)^2(d\underline{x}^3)^2\} \quad \ldots(32)$$

which is the metric tensor for spherical mass distribution and is known as the Schwarschild solution. (Adler 1965).

Case II : $\zeta = 0$

For $\zeta = 0$ eqn. (12) gives after using eqns. (7), (16), (18), (20) and (24)

$$AB = 0 \quad \ldots(33)$$

which implies that F given by eqn. (20) must take the following form

$$F = D\,\xi^{2n} \quad \ldots(34)$$

where D is a constant and $n = 1 \pm (c^2 + 1)^{1/2}$. Using the same procedure used before we find u^i from eqns. (13), (14) and (15)

$$u^1 = 0,\ u^2 = D\xi^{n-c},\ u^3 = iE\xi^{n+c}. \quad \ldots(35)$$

D and E are constants. Putting eqns. (35), (17), (16) and (7) in eqn. (1) we get

$$ds^2 = -\xi^2 dt^2 + (dx^1)^2 + \frac{(dx^2)^2}{D^2\,\xi^{-n-2c}} + \frac{(dx^3)^2}{E^2\,\xi^{-n+2c}} \quad \ldots(36)$$

where ξ can be easily calculated from equation (16). After rescaling the coordinates and making a coordinate transformation $x^1 = f(\underline{x}^1)$ such that $\xi = \exp(M\underline{x}^1)$ we can put the metric in a following form

$$ds^2 = -\exp(2M\underline{x}^1)dt^2 + \exp(2M\underline{x}^1 - 4n\underline{x}^1)(dx^1)^2 + \exp(2cM\underline{x}^1 - 2nM\underline{x}^1)(dx^2)^2 + \exp(-2cM\underline{x}^1 - 2nM\underline{x}^1)(d) \quad \ldots(37)$$

The above solution is actualy Weyl line mass solution (Trautman 1964).

We have now obtained all the possible solutions of vacuum field equations having the property as descibed earlier. Before closing we should mention, however that the method used in this paper could be extended to find solutions of the field equations which admit at least one hypersurface orthogonal killing vector without having any specific property.

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