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ANALOGUE OF BIRKHOFF'S THEOREM IN EINSTEIN-CARTAN THEORY FOR ZEL'DOVICH FLUID

S. B. KALYANSHETTI

Department of Mathematics, Vivekanand College, Kolhapur 416002, *M.S.*

AND

B. B. WAGHMODE

Department of Mathematics, Shivaji University, Kolhapur 416004

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In this paper we show that the analogue of Birkhoff's theorem exists in Einstein-Cartan theory for Zel'dovich fluid and extend this result to Sen-Dunn-Einstein-Cartan theory. In Sen-Dunn-Einstein-Cartan theory the Birkhoff's theorem for Zel'dovich fluid (i.e. $\overline{P} = \overline{\rho}$) is true if the scalar field x^0 is independent of time.

1. INTRODUCTION

Recently Sen and Dunn (1971) have formulated a new scalar-tensor theory of gravitation in a modified riemannian manifold. The scalar field is characterized by the function $x^0 = x^0(x^i)$, where x^i are coordinates in the four-dimensional Lyra manifold and the tensor field is identified with the metric tensor g_{ij} of the manifold.

Reddy (1973) has shown that analogue of Birkhoff's theorem exists in a scalar-tensor theory of gravitation proposed by Sen and Dunn. Krori and Nandy (1977) have shown that a sufficient condition for Birkhoff's theorem to hold for the Sen-Dunn and Ross tensor theories of gravitation is that the scalar field should be independent of time.

Kalyanshetti and Waghmode (1981) have shown that analogue of Birkhoff's theorem exists in Sen-Dunn theory for the perfect Zel'dovich fluid if the scalar field is independent of time. Prasanna (1975) following the work of Trautman obtained the three sets of solutions of a static fluid sphere in Einstein-Cartan theory by adopting Hehl's approach and Tolman's technique.

In this paper we show that analogue of Birkhoff's theorem exists in Einstein-Cartan theory for Zel'dovich fluid and extend this result to Sen-Dunn-Einstein-Cartan theory. In Sen-Dunn Einstein-Cartan theory the Birkhoff's theorem for Zel'dovich fluid exists if the scalar field x^0 is independent of time.

2. BIRKHOFF'S THEOREM FOR ZEL'DOVICH FLUID IN EINSTEIN-CARTAN THEORY

We consider a spherically symmetric metric in the form

$$ds^2 = -(e^{2\lambda}\, dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + e^{2\nu}\, dt^2 \qquad \ldots(2.1)$$

where $\lambda = \lambda(r, t)$, $\nu = \nu(r, t)$.

The field equations for the perfect fluid in Einstein-Cartan theory (Prasanna 1975) for the metric (2.1) are

$$e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + 16\pi^2 K^2 = 8\pi P \qquad \text{...(2.2)}$$

$$e^{-2\lambda}\left[\nu'' + \nu'^2 - \lambda'\nu' + \frac{1}{r}(\nu' - \lambda')\right] - e^{-2\nu}(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}) + 16\pi^2 K^2 = 8\pi P \qquad \text{...(2.3)}$$

$$e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} + 16\pi^2 K^2 = 8\pi\rho \qquad \text{...(2.4)}$$

$$-e^{-2\lambda}\frac{2\dot{\lambda}}{r} = 0 \qquad \text{...(2.5)}$$

$$e^{-2\nu}\frac{2\dot{\lambda}}{r} = 0$$

where K is the spin component and we have assumed that the spins of the particles composing the fluid are all aligned in the radial direction only.

Taking $\bar{p} = (p - 2\pi K^2)$ and $\bar{\rho} = (\rho - 2\pi K^2)$, the above field equations reduce to

$$e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi\bar{P} \qquad \text{...(2.6)}$$

$$e^{-2\lambda}\left[\nu'' + \nu'^2 - \lambda'\nu' + \frac{1}{r}(\nu' - \lambda')\right] - e^{-2\lambda}(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}) = 8\pi\bar{p} \qquad \text{...(2.7)}$$

$$e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\bar{\rho} \qquad \text{...(2.8)}$$

$$e^{-2\nu}\frac{2\dot{\lambda}}{r} = 0. \qquad \text{...(2.9)}$$

For the Zel'dovich fluid $\bar{p} = \bar{\rho}$.

Hence from eqns. (2.6) and (2.8) we get

$$\left(\frac{\lambda' - \nu'}{r} - \frac{1}{r^2}\right) + \frac{e^{2\lambda}}{r^2} = 0$$

i.e. $[r(\lambda' - \nu') - 1] + e^{2\lambda} = 0.$ ...(2.10)

From eqn. (2.9) we get $\dot{\lambda} = 0$ which implies that λ is independent of time i.e.

$\lambda = \lambda(r)$.

Differentiating eqn. (2.10) w.r.t. t we get

$$r(\dot{\lambda}' - \dot{\nu}') + 2e^{2\lambda}\dot{\lambda} = 0. \qquad \text{...(2.11)}$$

From this we get

$$\dot{\nu}' = 0. \qquad \ldots(2.12)$$

This implies that $\nu = f(t) + g(r)$. Introducing a time transformation $dt' = e^{f(t)/2}\, dt$ and dropping the primes we see that the metric (2.1) is static. Therefore ν is independent of time.

$$\nu = \nu(r).$$

Thus we see that for the perfect Zel'dovich fluid every spherically symmetric solution in Einstein-Cartan theory is static.

3. Birkhoff's Theorem in Sen-Dunn-Einstein-Cartan Theory for the perfect Zel'dovich Fluid

We consider the spherically symmetric line element in the form

$$ds^2 = -e^{2\lambda}\, dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{2\nu}\, dt^2 \qquad \ldots(3.1)$$

where $\lambda = \lambda(r, t)$, $\nu = \nu(r, t)$.

The field equations given by Sen and Dunn for the combined scalar and tensor fields are

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi G (x^0)^{-2}\, T_{ij} + \omega (x^0)^{-2} \Big(x^0{}_{,i}\, x^0{}_{,j} - \frac{1}{2} g_{ij}\, x^{0,k}\, x^0{}_{,k}\Big) \qquad \ldots(3.2)$$

where T_{ij} is the energy-momentum tensor and R_{ij} is the Ricci tensor. R is the usual Riemann curvature scalar and $\omega = 3/2$. The scalar field is characterized by the function $x^0 = x^0(x^i)$, where x^i are the coordinates in the four-dimensional Lyra's manifold. The tensor field is identified with the metric tensor g_{ij}. Here $x^0 = x^0(r, t)$.

The field equations of Sen and Dunn for the perfect fluid in Einstein-Cartan theory are

$$e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi\Big[P(x^0)^{-2} - 2\pi K^2\Big] - \frac{1}{2}\omega (x^0)^{-2}\Big[e^{-2\lambda}(x^{0\prime})^2 + e^{-2\nu}(\dot{x}^0)^2\Big] \qquad \ldots(3.3)$$

$$e^{-2\lambda}\Big[\nu'' + \nu'^2 - \lambda'\nu' + \frac{1}{r}(\nu' - \lambda')\Big] - e^{-2\lambda}(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}) = 8\pi\Big[P(x^0)^{-2} - 2\pi K^2\Big] + \frac{1}{2}\omega (x^0)^{-2}\Big[e^{-2\lambda}(x^{0\prime})^2 - e^{-2\nu}(\dot{x}^0)^2\Big] \qquad \ldots(3.4)$$

$$e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\Big[\rho(x^0)^{-2} - 2\pi K^2\Big] - \frac{1}{2}\omega (x^0)^{-2} \times \Big[e^{-2\lambda}(x^{0\prime})^2 + e^{-2\nu}(\dot{x}^0)^2\Big] \qquad \ldots(3.5)$$

$$\frac{2\dot{\lambda}}{r} = \omega\,(x^0)^{-2}\,(x^{0\prime})\,(\dot{x}^0) \qquad \ldots(3.6)$$

where we have assumed that the spins of the particles composing the fluid are all aligned in the r-direction only and we have set $G = 1$.

Taking $\overline{P} = [P\,(x^0)^{-2} - 2\pi K^2]$ and $\overline{\rho} = [\rho\,(x^0)^{-2} - 2\pi K^2]$ the above field equations can be written as

$$e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\,\pi\,\overline{P} - \frac{1}{2}\,\omega\,(x^0)^{-2}\Big[e^{-2\lambda}\,(x^{0\prime})^2 + e^{-2\nu}\,(\dot{x}^0)^2\Big] \qquad \ldots(3.7)$$

$$e^{-2\lambda}\left[\nu'' + \nu'^2 - \lambda'\nu' + \frac{1}{r}(\nu' - \lambda')\right] - e^{-2\nu}\,(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu})$$
$$= 8\,\pi\,\overline{P} + \frac{1}{2}\,\omega\,(x^0)^{-2}\Big[e^{-2\lambda}\,(x^{0\prime})^2 - e^{-2\nu}\,(\dot{x}^0)^2\Big] \qquad \ldots(3.8)$$

$$e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\,\pi\,\overline{\rho} - \frac{1}{2}\,\omega\,(x^0)^{-2}\Big[(x^{0\prime})^2\,e^{-2\lambda} + e^{-2\nu}\,(\dot{x}^0)^2\Big] \qquad \ldots(3.9)$$

For the perfect Zel'dovich fluid we have

$$\overline{P} = \overline{\rho}.$$

Therefore from eqns. (3.7) and (3.9) we get

$$\frac{\lambda' - \nu'}{r} - \frac{1}{r^2} + \frac{e^{2\lambda}}{r^2} = 0$$

i.e. $(\lambda' - \nu')\,r - 1 + e^{2\lambda} = 0.$ ...(3.10)

If the scalar field x^0 is a function of r alone then from the eqn. (3.6) we get

$$\dot{\lambda} = 0. \qquad \ldots(3.11)$$

This implies that λ is a function of r alone. Differentiating eqn. (3.10) w.r.t. t we get

$$r\,(\dot{\lambda}' - \dot{\nu}') + 2e^{2\lambda}\,\dot{\lambda} = 0 \qquad \ldots(3.12)$$

which implies that

$$\dot{\nu}' = 0.$$

This implies that $\nu = f(t) + g(r)$. Introducing a time transformation $dt' = e^{f(t)/2}\,dt$ and dropping the primes we see that the metric (3.1) is static.

Thus we see that for the perfect Zel'dovich fluid every spherically symmetric solution in Sen-Dunn-Einstein-Cartan theory is static if the scalar field x^0 is independent of time.

4. Conclusions

Analogue of Birkhoff's theorem exists for the perfect Zel'dovich fluid in Einstein-Cartan theory and in Sen-Dunn-Einstein-Cartan theory if the scalar field x^0 is independent of time.

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