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STABILITY AND INSTABILITY IN A DUST FILLED UNIVERSE

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In this paper we consider the line element $ds^2 = -e^{\xi}dt^2 + e^{\eta}dx^2 + R^2(dy^2 + dz^2)$, where ξ, η and R are the functions of t only. Using the tetrad formalism we obtain the equilibrium solutions for this dust model. Further the dynamical stability and instability of this model are discussed. The growth of density perturbation is investigated.

1. INTRODUCTION NOTATION, TETRAD FORMALISM AND FIELD EQUATIONS

Space time is represented as a four dimensional Riemannian geometry with the metric tensor g_{ij} of signature $(-, +, +, +)$. Covariant differentiation is indicated by a semi colon, and covariant differentiation along a world line by a dot over the variables.

The Einstein's field equations for dust filled cosmological models are (see Ellis 1967).

$$R_{ij} - \tfrac{1}{2}Rg_{ij} + \Lambda g_{ij} = T_{ij} \qquad \ldots(1.1)$$

$$T_{ij} = \rho U_i U_j, \qquad \ldots(1.2)$$

where Λ is the cosmological constant having dimensions $(\text{length})^{-2}$ of space curvature.

If (1.2) holds with $\rho > 0$ the matter content of the universe is called dust. The term Λ in the field equations to counteract the effect of gravitational attraction of matter U^a is the 4-velocity of matter. This vector is taken to be normalized, so that $U^a U_a = -1$.

In the units used $c = 1$ and $G = 1/8\pi$.

Latin indices run from 0 to 3; $i, j, k, \ldots$ are co-ordinate indices and $a, b, c\ldots$ are tetrad indices. Greek indices run over 1, 2, 3, the vector e_0 is time like, so $\{e_\nu\}$ are spacelike. Round brackets denote symmetrized indices, and square brackets denote skew-symmetrized indices.

The tensors $\dot{u}_j$, Θ, σ_{ij} and ω_{ij} are defined as

$$u_j = u_{i;j}\, u^i,$$

where velocity gradient $u_{i;j}$ may be further split up as

$$u_{i;j} = \omega_{ij} + \sigma_{ij} + \tfrac{1}{3}\Theta h_{ij} - \dot{u}_i u_j, \qquad \ldots(1.3)$$

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where $\Theta = \theta_{11} + \theta_{22} + \theta_{33}$ is expansion scalar.

$$\sigma_{ij} = u_{(i;j)} + \dot{u}_{(i}u_{j)} - \tfrac{1}{3}\theta h_{ij}, \qquad \ldots(1.4)$$

is the trace free tensor.

$$W_{ij} = U_{[i,j]} + \dot{U}_{[i}U_{j]}, \qquad \ldots(1.5)$$

is the vorticity tensor giving the rotation of the flow lines.

The Ricci rotation coefficients are (see Weatherburn 1966),

$$\Gamma_{abc} = e_a \nabla_b e_c = e_a^i \, e_{c_{i;j}} \, e_b^j.$$

The Lie derivative of e_b with respect to e_a is $[e_a, e_b]$,
where $[e_a, e_b] f = \partial_a(\partial_b f) - \partial_b(\partial_a f)$.

It is a vector with tetrad components $\gamma^c_{\cdot ab}$

$$[e_a, e_b] = \gamma^c_{\cdot ab}\, e_c$$

such as $\gamma^c_{\cdot ab} = \gamma_{\cdot [ab]}^{c}$.

The γ^c_{ab} and $\Gamma^c_{\cdot ab}$ are linearly dependent quantities,

$$\left.\begin{aligned} \gamma^c_{\cdot ab} &= \Gamma^c_{\cdot ab} - \Gamma^c_{\cdot ba} \\ \Gamma_{abc} &= \tfrac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca}). \end{aligned}\right\} \qquad \ldots(1.6)$$

The field equations can be written down in tetrad form as:

$$\begin{aligned} R_{bd} &= \partial_d \, \Gamma^c_{\cdot cb} - \partial_c \Gamma^c_{db} - \Gamma^c_{\cdot cg} \, \Gamma^g_{\cdot db} + \Gamma^g_{\cdot cb} \, \Gamma^c_{\cdot gd} \\ &= -(\tfrac{1}{2}\rho + \Lambda) h_{bd} - (\tfrac{1}{2}\rho - \Lambda) u_b u_d. \end{aligned} \qquad \ldots(1.7)$$

The contracted Bianchi identities for dust (1.2) give,

$$\dot{\rho} + \rho\Theta = 0 \qquad \ldots(1.8)$$

which is the mass conservation equation. The antisymmetry property of curvature tensor $R^i_{[jkl]} = 0$, is equivalent to the Jacobi identity,

$$\partial_{[d} \, \gamma^f_{\cdot bc]} + \gamma^f_{\cdot g[d} \, \gamma^g_{\cdot bc]} = 0. \qquad \ldots(1.9)$$

Now we make the first tetrad specialisation by choosing the time-like vector e_0 as the fluid flow vector u and e_1 lies along the vorticity vector W at each point so that,

$$U = e_0, \; u^a = \delta^a_0, \; \omega^i = \omega^i_1. \qquad \ldots(1.10)$$

Using suitable choice of co-ordinates and tetrad specialisation, we can express shear, expansion, vortity, acceleration etc. in tetrad form as follows (see Ellis 1967, Johri and Pathak 1976):

$$\gamma^{0}_{0\nu} = \dot{u}_{\nu} \text{ (acceleration components zero for dust)},$$

$$\gamma^{\nu}_{0\nu} = -\Theta_{\nu\nu} = -\Theta_{\nu} \qquad \text{...(1.11)}$$

$$\gamma^{0}_{\mu\nu} = -2\epsilon_{\mu\nu\sigma}\omega^{\sigma}, \; \epsilon_{\mu\nu\sigma} = \epsilon_{[\mu\nu\sigma]}, \; \epsilon_{123} = 1,$$

$$\gamma^{\mu}_{0\nu} = e_{\mu}.\dot{e}_{\nu} - \sigma_{\mu\nu} - \epsilon_{\mu\nu\sigma}\omega^{\sigma} \; (\mu \neq \nu),$$

$$\gamma^{\mu}_{\nu\mu} = e_{\nu}.\nabla_{\mu}e_{\nu} \text{ (no summation)},$$

and

$$\Theta = \theta_{11} + \theta_{22} + \theta_{33}$$
$$= (\gamma^{1}_{01} + \gamma^{2}_{02} + \gamma^{3}_{03}).$$

2. Equilibrium Solutions of the Model

Let us consider the line element as in Lightman *et al.* (1975) and Tolman (1934) as follows:

$$ds^2 = -e^{\xi}\, dt^2 + e^{\eta}dx + R^2(dy^2 + dz^2), \qquad \text{...(2.1)}$$

where ξ, η and R are functions of time t only.
The non vanishing tetrad components of the fundamental tensor in the line element (2.1) are given by,

$$e^{1}_{1} = \frac{1}{e^{\eta/2}}, \; e^{2}_{2} = e^{3}_{3} = \frac{1}{R}.$$

The components of $\gamma^{a}_{\cdot bc}$ are given by (see Ellis, 1967),

$$\left.\begin{aligned} &\theta_{\nu\nu} = -\gamma^{\nu}_{0\nu},\\ &\theta_{11} = \dot{\eta}/2,\; \theta_{22} = \theta_{33} = \dot{R}/R = \theta_0 \text{ (say)}\\ &\sigma_{\mu\nu} = -\tfrac{1}{2}\gamma^{\mu}_{0\nu} = 0(\mu < \nu),\; \omega = -\tfrac{1}{2}\gamma^{0}{}_{3} = 0\\ &\gamma^{2}_{12} = \gamma^{3}_{13} = \gamma^{3}_{23} = \gamma^{1}_{21} = \gamma^{1}_{31} = \gamma^{2}_{32} = \gamma^{2}_{12} = \gamma^{1}_{32} = 0. \end{aligned}\right\} \qquad \text{...(2.2)}$$

The field equations (1.7) for dust are,

$$\dot{\theta}_{11} + 2\dot{\theta}_0 + \theta^2_{11} + 2\theta^2_0 = \Lambda - \tfrac{1}{2}\rho \qquad \text{...(2.3a)}$$

$$\dot{\theta}_{11} + \theta^2_{11} + 2\theta_{11}\theta_0 = \Lambda + \tfrac{1}{2}\rho \qquad \text{...(2.3b)}$$

$$\dot{\theta}_0 + \theta_0\theta_{11} + 2\theta^2_0 = \Lambda + \tfrac{1}{2}\rho. \qquad \text{...(2.3c)}$$

Simplyfying the above equations as discussed by Ellis (1967), we get,

$$2\dot{\theta}_0 + 3\theta^2_0 = \Lambda \qquad \text{...(2.4)}$$

and $\dot{\theta}_{11} + \theta^2_{11} - \theta^2_0 = -\tfrac{1}{2}\rho.$ ...(2.4a)

Equilibrium solutions of the system of differential equations (2.4) and (2.4a) as method discussed by Braun (1978, see appendix), will be calculated as,

$$\theta_{11} = \pm\sqrt{\frac{1}{3}\Lambda},\ \theta_0 = \sqrt{\frac{1}{3}\Lambda - \frac{1}{2}\rho} \qquad ...(2.5)$$

Case I: STABILITY

For $\theta_{11}(t) = \sqrt{\frac{1}{3}\Lambda}$ and $\theta_0(t) = \sqrt{\frac{1}{3}\Lambda - \frac{1}{2}\rho}$,
eigen values at equilibrium state are,

$$\lambda_{11} = -\sqrt{\frac{1}{3}\Lambda - \frac{1}{2}\rho} \text{ and } \lambda_{22} = -\sqrt{\frac{1}{3}\Lambda}$$

If $\frac{1}{3}\Lambda > \frac{1}{2}\rho$ then eigen values are negative, therefore then the system is stable.

Case II: INSTABILITY

For $\theta_{11}(t) = \sqrt{\frac{1}{3}\Lambda}$ and $\theta_0(t) = \sqrt{\frac{1}{3}\Lambda - \frac{1}{2}\rho}$.
then the eigen values are,

$$\lambda_{11} = \sqrt{\frac{1}{3}\Lambda},\ \lambda_{22} = -\sqrt{\frac{1}{3}\Lambda - \frac{1}{2}\rho}$$

Here one eigen value is positive. therefore the system is unstable. From (2.5) we have,

$$\eta = 2\sqrt{\frac{1}{3}\Lambda}\,t \qquad ...(2.6)$$

$$R = \exp\left\{\left(\frac{1}{3}\Lambda - \frac{1}{2}\rho\right)^{1/2} t\right\} \qquad ...(2.7)$$

$$\text{and} \quad \theta_{11}^2 - \theta_0^2 = \tfrac{1}{2}\rho. \qquad ...(2.7a)$$

We can find a relation using (2.7) as,

$$\rho = \frac{2}{3}\Lambda - (\log R^{1/t})^2 \qquad ...(2.8)$$

From (2.8) we observe that density ρ continuously increases with time, since $(\log R^{1/t})^2$ is positive quantity.

3. THEORY OF PERTURBATION

We put the conditions for perturbation as stated in Johri and Pathak (1976). The new set of tetrads fixed to the perturbed model is of the form,

$$e_1^1 = \frac{1}{e^{\eta/2}} + \epsilon_{11}[x^0, x^\nu], e_2^1 = \pi_{12}[x^0, x^\nu],$$

$$e_2^2 = \frac{1}{R} + \epsilon_{22}[x^0, x^\nu],\ e_3^1 = \pi_{13}[x^0, x^\nu],$$

$$e_3^3 = \frac{1}{R} + \epsilon_{33}[x^0, x^\nu],\ e_3^2 = \pi_{23}[x^0, x^\nu],\ e_{30} = ye_3^3. \qquad ...(3.1)$$

Where $Y = y[x^2, x^3]$ is small of the order of perturbation terms. The other tetrad perturbations and the corresponding perturbations of $\gamma^{a}_{\cdot bc}$ from the background model to the perturbed one will be given by,

$$\delta e^1_1 = \epsilon_{11}, \ -\delta\theta_{11} = \partial_0(\epsilon_{11} e^{\eta/2}); \ \delta e^2_2 = \epsilon_{22}, \ -\delta\theta_{22} = \partial_0(\epsilon_{22} R),$$

$$\delta e^3_3 = \epsilon_{33}, \ -\delta\theta_{33} = \partial_0\ (\epsilon_{33} R),$$

$$\delta e^1_2 = e^1_2 = \pi_{12}, \ -2\sigma_{12} = \frac{e^{\eta/2}}{R}\partial_0(\pi_{12} R),$$

$$\delta e^1_3 = e^1_3 = \pi_{13}, \ -2\sigma_{13} = \frac{e^{\eta/2}}{R}\partial_0(\pi_{13} R),$$

$$\delta e^2_3 = e^2_3 = \pi_{23}, \ -2\sigma_{23} = \partial_0(\pi_{23} R).$$

$$\gamma^2_{12} = \frac{R}{e^{\eta/2}}\partial_1(\epsilon_{22}), \ \gamma^2_{13} = \frac{R}{e^{\eta/2}}\partial_1\ (\pi_{23}),$$

$$\gamma^3_{13} = \frac{R}{e^{\eta/2}}\partial_1\ (\epsilon_{33}), \ \gamma^1_{32} = \frac{e^{\eta/2}}{R}\partial_3(\pi_{12}) - \partial_2(\pi_{13}),$$

$$\gamma^3_{23} = \partial_2(\epsilon_{33}), \ \gamma^1_{21} = \frac{e^{\eta/2}}{R}\partial_2(\epsilon_{11}) - \partial_1(\pi_{12}),$$

$$-2\omega = \gamma^0_{23} = \frac{1}{R^2}\partial_2 y\ ,$$

$$\gamma^1_{31} = \frac{-y\dot{\eta}}{2R} + \frac{e^{\eta/2}}{R}\partial_3(\epsilon_{33}) - \partial_1(\pi_{13}),$$

$$\gamma^2_{32} = \frac{-y\dot{\eta}}{R^2} + \partial_3(\epsilon_{22}) - \partial_2(\pi_{23}). \qquad \ldots(3.2)$$

Simplified first order in perturbed field equations will be given as follows,

$$\partial_0[\delta\theta_{11} + \delta\theta_{22} + \delta\theta_{33}] + 2\theta_{11}\delta\theta_{11} + 2\theta_0[\delta\theta_{22} + \delta\theta_{33}] = -\tfrac{1}{2}\delta\rho,$$

$$\partial_0[\partial_1(\epsilon_{22} + \epsilon_{33})R] = \frac{e^{\eta/2}}{2R}\partial_0[\partial_2(\pi_{12} + \pi_{13})\ R], \qquad \ldots(00)$$

$$\partial_3\omega = \partial_0[\partial_2(\epsilon_{33} + \epsilon_{11})R] - \tfrac{1}{2}\partial_0[\partial_3(\pi_{23}R)] - \frac{e^{\eta/2}}{2R}\partial_0(\pi_{12}R), \qquad \ldots(01)$$

$$-\partial_3\omega = \partial_0[\partial_3(\epsilon_{11}e^{\eta/2} + \epsilon_{22}R)] - \frac{e^{\eta/2}}{2R}\partial_0\,\partial_1\,(\pi_{13}R) - \tfrac{1}{2}\partial_0\,\partial_2\,(\pi_{23}R), \qquad \ldots(02)$$

$$\partial_0(\delta\theta_{11}) + \frac{\dot{\eta}}{2}(2\delta\theta_{11} + \delta\theta_{22} + \delta\theta_{33}) + \frac{2\dot{R}}{R}\delta\theta_{11} + \frac{R}{e^{\eta/2}}[\partial_1\partial_1(\epsilon_{22} + \epsilon_{33})] \qquad \ldots(03)$$

$$+ \frac{e^{\eta/2}}{R}[(\partial_2\partial_2 + \partial_3\partial_3)\epsilon_{11}] - \partial_1\partial_2(\pi_{12}) - \partial_1\partial_3(\pi_{13}) - \frac{\dot{\eta}}{2R}\partial_3 y = \tfrac{1}{2}\delta\rho, \qquad \ldots(11)$$

$$\partial_0(\delta\theta_{22}) + \frac{\dot{\eta}}{2}\delta\theta_{22} + \frac{\dot{R}}{R}(\delta\theta_{11} + 3\theta_{22} + \delta\theta_{33}) + \partial_2\partial_2\left(\frac{e^{\eta/2}}{R}\epsilon_{11} + \epsilon_{33}\right)$$

$$+\left[\frac{R}{e^{\eta/2}}\partial_1\partial_1 + \partial_3\partial_3\right]\epsilon_{22} - \partial_1\partial_2\pi_{12} - \partial_2\partial_3\pi_{23} - \frac{\dot{R}}{R^2}\partial_3 y = \tfrac{1}{2}\delta\rho, \qquad \ldots(22)$$

$$\partial_0(\delta\theta_{33}) + \frac{\dot{\eta}}{2}\delta\theta_{33} + \frac{\dot{R}}{R}(\delta\theta_{11} + \delta\theta_{22} + 3\delta\theta_{33}) + \partial_3\partial_3\left[\frac{e^{\eta/2}}{R}\epsilon_{11} + \epsilon_{22}\right]$$
$$+\left[\frac{R}{e^{\eta/2}}\partial_1\partial_1 + \partial_2\partial_2\right]\epsilon_{33} - \partial_1\partial_3\pi_{13} - \partial_2\partial_3(\pi_{23}) - \frac{\dot{\eta}}{2R}\partial_3 y - \frac{\dot{R}}{R^2}\partial_3 y = \tfrac{1}{2}\delta\rho, \quad \ldots(33)$$

$$\partial_0(\sigma_{12}) + \sigma_{12}\left(\frac{\dot{R}}{R} + \dot{\eta}\right) = \frac{R}{2e^{\eta/2}}[\partial_1\partial_2\pi_{23} - 2\partial_1\partial_2\epsilon_{33}] + \tfrac{1}{2}[\partial_2\partial_3\pi_{13} - \frac{e^{\eta/2}}{R}\partial_3\partial_3\pi_{12}], \quad \ldots(12)$$

$$\partial_0(\sigma_{23}) + \sigma_{23}\left(\frac{\dot{\eta}}{2} + \frac{2\dot{R}}{R}\right) = \left(1 - \frac{e^{\eta/2}}{2R}\right)\partial_1\partial_3\pi_{13} + \frac{e^{\eta/2}}{2R}[\partial_1\partial_2\pi_{13} - 2\partial_2\partial_3\epsilon_{11}]$$
$$- \frac{-R}{2^{\eta/2}}\partial_1\partial_1\pi_{23}, \quad \ldots(23)$$

$$\partial_0(\sigma_{13}) + \sigma_{13}\left(\frac{\dot{R}}{R} + \dot{\eta}\right) = \left(\frac{1-R}{2e^{\eta/2}}\right)\partial_1\partial_2\pi_{23} + \frac{e^{\eta/2}}{2R}[\partial_2\partial_3\pi_{12} - \partial_2\partial_3\pi_{13}]$$
$$- \partial_1\partial_3\epsilon_{22}. \quad \ldots(31)$$

4. Growth of Density Perturbations

From field equation (1.7) and (00), and taking the values of θ_{11} and θ_0 and from (2.5) we find,

$$\theta_{11} = \sqrt{\frac{1}{3}\Lambda} \text{ and } \theta_0 = \sqrt{\frac{1}{3}\Lambda - \frac{1}{2}\rho}.$$

Hence $\theta_0^2 = \frac{1}{3}\Lambda - \frac{1}{2}\rho$ and $\theta_{11}^2 = \frac{1}{3}\Lambda$, ...(4.1)

$$\therefore\ 2\theta_0\delta\theta_0 = -\frac{1}{2}\delta\rho,\ 2\theta_{11}\delta\theta_{11} = 0,$$

$$\therefore \partial_0\left(\partial_0\frac{\delta\rho}{\rho}\right) + 0 - \frac{1}{2}\delta\rho + \frac{1}{2}\delta\rho = 0.$$

Putting $\frac{\delta\rho}{\rho} = \mu$ and denote $\partial_0^2 \equiv D^2$, we get,

$$D^2\mu = 0.$$

Integrating this equation we get,

$$\mu = \frac{\delta\rho}{\rho} = Kt + C_1$$

from this relation growth of density perturbation is directly proportional to the time.

5. Conclusion

If we put $\xi(t) = 0$, $e^{\eta(t)} = R^2(t)$, then this model will be reduced to Einstein de-Sitter model which was discussed by Johri and Pathak (1976).

Appendix

Problem

Find all possible equilibrium solutions of the system of differential equations,

$$\frac{d\theta_0}{dt} = 1 - \theta_0\theta_1, \frac{d\theta_1}{dt} = \theta_0 - \theta_1^3 \quad \ldots(A1)$$

and hence (if possible) discuss the system for stability.

Solution

From above equation (1), when the system is equilibrium then, $1-\theta_0\theta_1=0$ and $\theta_0 - \theta_1^3 = 0$ this imply that

$$\theta_0 = 1, \theta_1 = 1 \text{ or } \theta_0 = 1, \theta_1 = -1.$$

Hence $\theta_0(t) = 1$, $\theta_1(t) = 1$ and $\theta_0(t) = -1$, $\theta_1(t) = -1$ are the only solutions of (1).

Case I—For $\theta_0(t) = 1$, $\theta_1^{(t)} = 1$; set $P = \theta_0 - 1$, $Q = \theta_1 - 1$. Then

$$\frac{dP}{dt} = \frac{d\theta_0}{dt} = 1 - (1 + P)(1 + Q) = P - Q - PQ$$

$$\frac{dQ}{dt} = \frac{d\theta_{11}}{dt} = (1 + P) - (1 + Q)^3 = P - 3Q - 3Q^2 - Q^3. \quad \ldots(A2)$$

The above system of equations (A2) can be written in the matrix form as follows,

$$\frac{d\begin{bmatrix} P \\ Q \end{bmatrix}}{dt} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} PQ \\ 3Q^2+Q^3 \end{bmatrix}.$$

Now consider the matrix, $\begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ which has a single eigen value $\lambda = -2$,

Since det $\begin{bmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{bmatrix}$.

Hence, we conclude that the equilibrium solution asymptotically stable.

Case II—For $\theta_0(t) = -1$, $\theta_1(t) = -1$ Set $P = \theta_0 + 1$, $Q = \theta_1 + 1$, then we get the eigenvalues as above,

$\lambda_1 = -1 - \sqrt{5}$, which is negative,

and $\lambda_2 = -1 + \sqrt{5}$, which is positive.

Therefore the equilibrium solution $\theta_0(t) = -1$,

$\theta_1(t) = -1$ of (1) is unstable.

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