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ON GOL'DBERG ORDER AND GOL'DBERG TYPE OF AN ENTIRE FUNCTION OF SEVERAL COMPLEX VARIABLES REPRESENTED BY MULTIPLE DIRICHLET SERIES

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The object of this paper is to obtain a necessary and sufficient condition for a multiple Dirichlet series to be entire. We then characterize the order and type of an entire function of several complex variables represented by multiple Dirichlet series and express them in terms of its coefficients and exponents.

1. INTRODUCTION AND NOTATIONS

Throughout the paper we denote complex and real n-space by $\mathbb{C}^n$ and R^n respectively. We indicate the elements $(s_1, s_2, \ldots, s_n)$, $(\text{Re } s_1, \text{Re } s_2, \ldots, \text{Re } s_n)$, $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ $(m_1, m_2, \ldots, m_n)$ etc. of $\mathbb{C}^n$ by their corresponding unsuffixed symbols s, $\text{Re } s$, σ, m etc. and make use of the standard notations of the single variable which are easy to understand from the context. For $x, y \in \mathbb{C}^n$, we define $x = (x_1, x_2, \ldots, x_n)$. $y = (y_1, y_2, \ldots, y_n)$, $xy = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$ $\| x \| = x_1 + x_2 + \ldots + x_n$, $x + r = (x_1 + r, x_2 + r, \ldots, x_n + r)$, for $r \in R$. Also for $x, y \in R^n$, we say that *i*) $x \leqslant y \Leftrightarrow x_j \leqslant y_j$, $j = 1, 2, \ldots n$; ii) $x < y \Leftrightarrow x \leqslant y$ but $x \neq y$; iii) $x \ll y \Leftrightarrow x_j < y_j$, $j = 1, 2, \ldots n$.

Consider the multiple Dirichlet series

$$f(s_1, s_2, \ldots s_n) = \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} a_{m_1, m_2, \ldots m_n} \exp\, (s_1 \lambda_{1 m_1} + s_2 \lambda_{2 m_2} + \ldots + s_n \lambda_{n\, m_n})$$

$$\text{i.e. } f(s) = \sum_{m=1}^{\infty} a_m \exp \| s\, \lambda_{n m_n} \| , \; (s_j = \sigma_j + i\tau_j, \; j = 1, 2, \ldots n) \qquad \ldots(1.1)$$

where $a_m \in \mathbb{C}$, $\lambda_{n m_n}$ denotes the real-tuple $(\lambda_{1 m_1}, \lambda_{2 m_2}, \ldots, \lambda_{n m_n})$;

$$0 \leqslant \lambda_{p1} < \lambda_{p2} < \ldots < \lambda_{pk} \to \infty \text{ as } k \to \infty, \quad \text{for } p = 1, 2, \ldots n.$$

Janusauskas (1977) had shown that if $\exists$ a tuple $p > \bar{0} = (0, 0, \ldots 0)$ s t

$$\limsup_{\| m \| \to \infty} \frac{\sum_{k=1}^{n} \log m_k}{\| p\, \lambda_{n m_n} \|} = 0 \qquad \ldots(1.2)$$

then the domain of absolute convergence of the series (1.1) coincides with its domain of convergence.

In our discussion we shall consider only those multiple Dirichlet series whose co-ordinates of associated abscissas of convergence will be all finite or infinite, but not both.

2. ENTIRE FUNCTION

Theorem 1—The necessary and the sufficient condition that the series (1.1) satisfying (1 2) to be entire is that

$$\lim_{\|m\| \to \infty} \frac{\log |a_m|}{\|\lambda_{nm_n}\|} = -\infty. \qquad \ldots(2.1)$$

PROOF : Let $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ be an n-tuple of associated abscissas of absolute convergence of (1.1) satisfying (1.2). Then (Janusauskas 1977)

$$\limsup_{\|m\| \to \infty} \frac{\log |a_m|}{-(\|\Delta \lambda_{nm_n}\|)} = 1. \qquad \ldots(2.2)$$

Let (1.1) be entire. Then $\Delta_1 > M$, $\Delta_2 > M$, ..., $\Delta_n > M$ for any large M. From (2.2), $\log |a_m| < -M(1+\epsilon)\|\lambda_{nm_n}\|$, for $\|m\| > N$.

So, $$\lim_{\|m\| \to \infty} \frac{\log |a_m|}{\|\lambda_{nm_n}\|} = -\infty.$$

Next we assume that (1.1) satisfying (1.2) having Δ as an n-tuple of associated abscissas of absolute convergence satisfies (2.1).

If possible, let all $\Delta_1, \Delta_2, \ldots, \Delta_n$ be finite so that $\Delta_1 < M, \Delta_2 < M, \ldots \Delta_n < M$. Let $s' \in P$ where $P = \{s : s \in \mathbb{C}^n, \operatorname{Re} s = \sigma \gg \Delta\}$.
From (2.1), $|a_m| < \exp\{-M \|\lambda_{nm_n}\|\}$, for $\|m\| > N_0$.

Let $0 < \epsilon_j < \dfrac{M - \Delta_j}{3}$, $j = 1, 2, \ldots n$. Also for $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, we set that $\operatorname{Re} s' = \sigma' = \Delta + \epsilon$. Then

$$|a_m \exp \|s' \lambda_{nm_n}\|| = |a_m| \exp \|\sigma' \lambda_{nm_n}\|$$

$$< \exp[-\{\sum_{j=1}^{n} \lambda_{jm_j} (M - \Delta_j - \epsilon_j)\}], \text{ for } \|m\| > N_0$$

$$< \exp\{-2 \|\epsilon \lambda_{nm_n}\|\}.$$

More over, (1.2) implies that for any $\epsilon' > 0$

$$\log (\sum_{j=1}^{n} m_j) < \epsilon' \|p \lambda_{nm_n}\|, \text{ for } \|m\| > N_0.$$

We choose ϵ' so small that $\epsilon' p \ll \epsilon$. Thus

$$\exp\{-2\,\|\epsilon\,\lambda_{nm_n}\|\} < \frac{1}{\prod_{j=1}^{n} m_j^2}, \text{ for } \|m\| > N_0.$$

Hence, $$| a_m \exp \| s'\,\lambda_{nm_n}\| | < \frac{1}{\prod_{j=1}^{n} m_j^2}, \text{ for } \|m\|\ N_0.$$

But the series $\sum_{m=1}^{\infty} \frac{1}{\prod_{j=1}^{n} m_j^2}$ is convergent and hence $\sum_{m=1}^{\infty} | a_m \exp \| s'\,\lambda_{nm_n}\| |$ is convergent which is a contradiction. Hence all $\Delta_1, \Delta_2, \ldots, \Delta_n$ are infinite i.e. the series (1.1) represents an entire function.

3. The Maximum Modulus and the Maximum Term

Throughout this paper F stands for the family of all multiple Dirichlet series of the form (1.1) satisfying (1.2) and (2.1). Then $f \in F$ denotes an entire function over $\mathbb{C}^n$.

For given $l \in R^n$, we define the poly half-plane D as $D = \{s \,;\, s \in \mathbb{C}^n,\ \text{Re}\, s = \sigma \leqslant l\}$. Then the region $D + r$ depending on the parameter $r \in R$ is defined as $D + r = \{s + r \,;\, s \in D\}$.

For any $f \in F$, we define the maximum modulus $M_{f,D}(r)$ w.r.t the region D where $r \in R$ as ;

$$M_{f,D}(r) = \sup\{\, | f(s) | : s \in D + r\}.$$

Also the maximum term $\mu_f = \mu_f(\sigma)$ at $\sigma \in R^n$ is defined by

$$\mu_f(\sigma) = \sup_{m \in N^n} \{\, | a_m | \exp \| \sigma\,\lambda_{nm_n} \| \,\}$$

where N is the set of all positive integers.

Theorem 2—Let $f \in F$. Then

$$\mu_f(l + r) \leqslant M_{f,D}(r) \leqslant K\,\mu_f(l + r + \epsilon) \qquad \ldots(3.1)$$

where K is a positive constant depending on $\epsilon \in R_+$.

Proof : Corresponding to $\epsilon' > 0$, $\exists\ N_0(\epsilon')$ s. t

$$2 \log \prod_{j=1}^{n} m_j < \epsilon\,\| \lambda_{nm_n} \|, \text{ for } \|m\| > N_0 \qquad \ldots(3.2)$$

where $\epsilon' p_1 < \epsilon$, $\epsilon' p_2 < \epsilon, \ldots, \epsilon' p_n < \epsilon$.

The inequality (3.2) is the consequence of (1.2)

By using Cauchy's inequality, we have

$$\mu_f(l + r) \leqslant M_{f,D}(r) \leqslant \Big[\sum_{\|m\| = n}^{N_0} + \sum_{\|m\| = N_0 + 1}^{\infty}\Big] | a_m | \exp \| (l + r)\,\lambda_{nm_n} \|$$

$$= S_1 + S_2 \text{ (say)}.$$

Now $S_1 \leqslant K_1\,\mu_f(l + r)$, K_1 being the no. of terms in S_1.

$$\text{Also, } S_2 = \sum_{\|m\| = N_0+1}^{\infty} | a_m | \exp \| (l + r + \epsilon) \lambda_{nm_n} \| \exp \{ - \epsilon \| \lambda_{nm_n} \| \}$$

$$\leqslant \mu_f (l + r + \epsilon) \sum_{\|m\| = N_0+1}^{\infty} \frac{1}{\prod_{j=1}^{n} m_j^2}, \text{ [using 3 2]}$$

$$\leqslant K_2 \mu_f (l + r + \epsilon).$$

Therefore, $\mu_f (l + r) \leqslant M_{f,D}(r) \leqslant K_1 \mu_f (l + r) + K_2 \mu_f (l + r + \epsilon)$

$$\leqslant K \mu_f (l + r + \epsilon).$$

4. The Order and Type of an Entire Function

We now define the order and the type of the function $f \in F$ based on the concept of A. A. Gol'dberg.

Let $f \in F$ and $S_f \subset R$ be the set of points $\alpha \in R$ s.t for every $\alpha \in S_f$, $\exists$ a $r_0 \in R$ s.t.

$\log M_{f,D}(r) \leqslant e^{r\alpha}$, for $r \geqslant r_0$.

The closure $\overline{S}_f$ of the set S_f is called the order set of f. The infimum of the set S_f is called the Gol'dberg order $\rho(D)$ of f w.r.t. the region D. We say that f is of infinite or finite order according as S_f is empty or non-empty.

Next, for the Gol'dberg order $\rho(D) = \rho > 0$, we denote $T_f(\rho)$ the set of all $T \in R$ s.t.

$\log M_{f,D}(r) \leqslant T e^{r\rho}$, for $r \geqslant r_0$.

The closure $\overline{T_f(\rho)}$ of the set $T_f(\rho)$ is called the type set of f and the infimum of the set $T_f(\rho)$ is called the Gol'dberg type $\tau(D)$ of f corresponding to ρ w.r.t. D. As before we say that f is of infinite or finite type according as $T_f(\rho)$ is empty or non-empty and the function f is of maximal, minimal, or mean type w.r.t. D according as $\tau(D) = \infty$, $\tau(D) = 0$, $0 < \tau(D) < \infty$.

Lemma 1—Let $f \in F$ be of Gol'dberg order $\rho(D)$ and Gol'dberg type $\tau(D)$. Then

$$\rho(D) = \limsup_{r \to \infty} \frac{\log \log M_{f,D}(r)}{r}$$

$$\tau(D) = \limsup_{r \to \infty} \frac{\log M_{f,D}(r)}{e^{r\rho(D)}}, \text{ if } \rho(D) > 0.$$

The above two are the direct consequences of the definition.

Theorem 3—Let $f \in F$. Then for any $K \in R$

i) $\rho(D + K) = \rho(D)$

ii) If $\rho(D) > 0$, then $\tau(D + K) = e^{K\rho} \tau(D)$ where $\rho = \rho(D)$.

Proof : Let $K \in R$, then from Lemma 1, we get

$$\text{(i) } \rho(D) = \limsup_{r \to \infty} \frac{\log \log M_{f,D}(r + K)}{r + K} = \limsup_{r \to \infty} \frac{\log \log M_{f,D+K}(r)}{r} \times \frac{r}{K + r} = \rho(D + K)$$

$$\text{(ii) } \tau(D) = \limsup_{r \to \infty} \frac{\log M_{f,D}(r+K)}{e^{(r+K)\rho}} = \limsup_{r \to \infty} \frac{\log M_{f,D+K}(r)}{e^{K\rho} \cdot e^{r\rho}}$$
$$= \frac{1}{e^{K\rho}} \tau(D+K).$$

Remark : From Theorem 3, it is clear that the Gol'dberg order of $f \in F$ is independent of the choice of the poly half plane D, but in the case of Gol'dberg type $\tau(D)$, it is not true.

We now establish the relations between the order, the type and the coefficients of $f \in F$.

For this purpose, the following lemma is required.

Lemma 2—Let $f \in F$ and for $\alpha \in R_+$, $T \in R_+$, $\exists$ a $r_0 \in R_+$ s.t.

$$\log M_{f,D}(r) \leqslant T e^{r\alpha}, \text{ for } r \geqslant r_0. \qquad \ldots(4.1)$$

$$\text{then } |a_m| \phi_D(m) \leqslant \left(\frac{eT\alpha}{\|\lambda_{n,m_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}}, \text{ for } \|m\| > N_0 \qquad \ldots(4.2)$$

and conversely, where $\phi_D(m) = \sup_{s \in D} |\exp \| s\lambda_{nm_n} \|| = \exp \| l\lambda_{nm_n} \|$.

PROOF : Suppose that(4.1) holds for $f \in F$. By Cauchy's inequality

$$|a_m| \exp \| (l+r)\lambda_{nm_n} \| \leqslant \sup \{ |f(s')| : s' \in D + r \}$$

$$\text{i.e. } |a_m| \phi_D(m) \leqslant \frac{M_{f,D}(r)}{\exp \| r\lambda_{nm_n} \|} \leqslant \frac{\exp(Te^{r\alpha})}{\exp \| r\lambda_{nm_n} \|}, \text{ for } r \geqslant r_0.$$

The R.H.S. attains its maximum value $\left(\frac{eT\alpha}{\|\lambda_{nm_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}}$ at $r = \frac{1}{\alpha} \log \frac{\|\lambda_{nm_n}\|}{T\alpha}$. Then $\exists$ an integer $N > 0$ s.t. $\frac{1}{\alpha} \log \frac{\|\lambda_{nm_n}\|}{T\alpha} \geqslant r_0$, for $\|m\| > N$.

Hence

$$|a_m| \phi_D(m) \leqslant \left(\frac{eT\alpha}{\|\lambda_{nm_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}}, \text{ for } \|m\| > N.$$

Next suppose that (4.2) holds for $\|m\| > N$. Then

$$M_{f,D}(r) = \sup_{s' \in D+r} \left| \sum_{m=1}^{\infty} a_m \exp \| s'\lambda_{nm_n} \| \right|$$
$$\leqslant \sum_{m=1}^{\infty} |a_m| \phi_D(m) \exp \| r\lambda_{nm_n} \| = \sum_{\|m\|=n}^{N} + \sum_{\|m\|=N+1}^{\infty}$$
$$\leqslant K_1 \exp \sum_{J=1}^{n} \{(l_j + r)\lambda_{J(N-n+1)}\}$$

(*equation continued on p.* 1226)

$$+ \sum_{\|m\| = N+1}^{\infty} \left(\frac{eT\alpha}{\|\lambda_{nm_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}} \cdot \exp \| r\, \lambda_{nm_n} \|$$

we set $P = e\alpha\, (T + \epsilon)\, e^{r\alpha}$, then for $\| \lambda_{nm_n} \| \leqslant P$,

$$\left(\frac{eT\alpha}{\|\lambda_{nm_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}} . \exp \| r\, \lambda_{nm_n} \| \leqslant \exp\, (T\, e^{r\alpha}), \text{ and for } \| \lambda_{nm_n} \| > P$$

$$\left(\frac{eT\alpha}{\|\lambda_{n\, m_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}} . \exp \| r\, \lambda_{n\, m_n} \| < \left(\frac{T}{T + \epsilon} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}}$$

Consequently,
$$\sum_{\|m\| = N+1}^{\infty} \left(\frac{eT\alpha}{\|\lambda_{n\, m_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\alpha}} . \exp \| r\, \lambda_{nm_n} \|$$
$$= \sum_{\|\lambda_{nm_n}\| \leqslant P} + \sum_{\|\lambda_{nm_n}\| > P} \leqslant K_2 \exp\, (Te^{r\alpha}) + K_3$$

where K_1, K_2 and K_3 are finite positive constants.

Hence, $M_{f,\, D}\, (r) \leqslant K_1 \exp \sum_{j=1}^{n} \{(l_j + r)\, \lambda_{j(N-n+1)}\} + K_2 \exp\, (T\, e^{r\alpha}) + K_3$

$\leqslant K_2 \exp\, (T\, e^{r\alpha})\, [1 + O\,(1)]$, O represents small O.

So, $\log M_{f,\, D}\, (r) \leqslant T\, e^{r\alpha}$, for $r \geqslant r_0$.

Theorem 4—$f \in F$ is of Gol'dberg order ρ iff

$$\rho = \rho\,(D) = \limsup_{\|m\| \to \infty} \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{-\log \{\, |\, a_m\, |\, \phi_D\, (m)\}} \qquad \ldots(4.3)$$

where $\phi_D\,(m) = \sup_{s \in D} |\, \exp \| s\, \lambda_{nm_n} \| \,|$.

PROOF: Let $f \in F$ be of Gol'dberg order ρ. Then for given $\epsilon > 0$, $\exists$ a $r_0(\epsilon) \in R_+$ s.t. $\log M_{f,\, D}\, (r) \leqslant e^{r(\rho+\epsilon)}$, for $r \geqslant r_0(\epsilon)$

From lemma 2, $|\, a_m\, |\, \phi_D\, (m) \leqslant \left(\frac{e\,(\rho + \epsilon)}{\|\lambda_{nm_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\rho+\epsilon}}$, for $\|m\| > N_0\,(\epsilon)$

Since $\left\{ |\, a_m\, |\, \phi_D\,(m) \right\}^{\frac{1}{\|\lambda_{nm_n}\|}} \to 0$ as $\| m \| \to \infty$, we find that

$$-\log \left\{ |\, a_m\, |\, \phi_D\,(m) \right\} \geqslant \frac{\|\lambda_{nm_n}\|}{\rho + \epsilon} . \log \|\lambda_{nm_n}\| + \chi\,(m)$$

where the function χ is s.t. $\dfrac{\chi\,(m)}{\log \{\, |\, a_m\, |\, \phi_D\,(m)\}} \to 0$ as $\| m \| \to \infty$.

Therefore, $\limsup_{\|m\| \to \infty} \dfrac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{-\log\{|a_m| \phi_D(m)\}} \leqslant \rho.$

Let, if possible,

$$\limsup_{\|m\| \to \infty} \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{-\log\{|a_m| \phi_D(m)\}} = \delta\rho < \rho \text{, where } 0 < \delta < 1.$$

Then for $\epsilon > 0$, $\exists$ an integer $N(\epsilon)$ s.t.

$$|a_m| \phi_D(m) < \left(\frac{1}{\|\lambda_{nm_n}\|}\right)^{\frac{\|\lambda_{nm_n}\|}{\rho\delta + \epsilon}}, \text{ for } \|m\| > N(\epsilon).$$

From lemma 2, $\log M_{f,D}(r) \leqslant \dfrac{\exp(r(\rho\delta + \epsilon))}{e(\rho\delta + \epsilon)}$, for $r \geqslant r_0$

$$< \exp(r(\rho\delta + 2\epsilon)), \text{ for } r \geqslant r_0'$$

which implies that $\rho\delta + 2\epsilon \in S_f$ and hence $\rho \in \text{int } S_f$. This contradiction proves the necessary condition.

Proof of the sufficient condition: It follows from (4.3) that for sufficiently large values of $\|m\|$, say $\|m\| > N$

$$|a_m| \phi_D(m) \leqslant \left(\frac{1}{\|\lambda_{nm_n}\|}\right)^{\frac{\|\lambda_{nm_n}\|}{\rho + \epsilon}}.$$

From lemma 2, $\log M_{f,D}(r) \leqslant \dfrac{\exp(r(\rho + \epsilon))}{e(\rho + \epsilon)}$, for $r \geqslant r_0$

$$\leqslant \exp(r(\rho + 2\epsilon)), \text{ for } r \geqslant r_0'$$

which shows that $\rho + 2\epsilon \in S_f$. ...(4.4)

Again from (4.3), it follows that $\exists$ a sequence $\{\|m^{(p)}\|\}$ with $\|m^{(p)}\| \to \infty$ as $p \to \infty$ for which

$$\log\{|a_m| \phi_D(m)\} > \frac{-\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{\rho - \epsilon},$$

$$\text{for } m_j = m_j^{(p)}\ j = 1, 2, \dots n.$$

Hence and from the relation (3.1), it follows that

$$\log M_{f,D}(r) > r\|\lambda_{nm_n}\| - \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{\rho - \epsilon},$$

$$\text{for } m_j = m_j^{(p)}, j = 1, 2, \dots, n \qquad \dots(4.5)$$

For sufficiently large p, say $p > p_0$, we have $r = r_p = \dfrac{\log \|e\, \lambda_{nm_n^{(p)}}\|}{\rho - \epsilon}$ and substituting the value of r_p in (4.5), we obtain

$$\log M_{f,D}(r) > \frac{\exp(r(\rho-\epsilon))}{\rho-\epsilon}, \text{ for } r = r_p$$

$$\not> \exp(r(\rho - 2\epsilon)), \text{ for } r = r_p \to \infty \text{ as } p \to \infty$$

which shows that $\rho - 2\epsilon \notin S_f$. ...(4.6)

from (4.4) and (4.6), it follows that ρ is the order of f.

Theorem 5— $f \in F$ is of order $\rho = \rho(D) \in R_+$ and type $\tau(D) \geqslant 0$ iff

$$\tau(D) = \frac{1}{e\rho} \limsup_{\|m\| \to \infty} \left[\|\lambda_{nm_n}\| \{|a_m| \phi_D(m)\}^{\frac{\rho}{\|\lambda_{nm_n}\|}} \right] \quad ...(4.7)$$

where $\phi_D(m) = \sup_{s \in D} |\exp \|s\, \lambda_{n\, m_n}\||$.

Proof : Let $f \in F$ be of order $\rho = \rho(D) \in R_+$ and type $\tau(D) \geqslant 0$. Then for given $\epsilon > 0$, $\exists$ a number $r_0 \in R_+$ s.t

$$\log M_{f,D}(r) \leqslant (\tau_1 + \epsilon)\, e^{r\rho}, \text{ for } r \geqslant r_0 \text{ and } \tau_1 = \tau(D).$$

From Lemma 2, $|a_m| \phi_D(m) \leqslant \left(\frac{e\rho(\tau_1+\epsilon)}{\|\lambda_{nm_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\rho}}$, for $\|m\| > N(\epsilon)$.

Thus, $\frac{1}{e\rho} \limsup_{\|m\| \to \infty} \left[\|\lambda_{n\, m_n}\| \{|a_m| \phi_D(m)\}^{\frac{\rho}{\|\lambda_{nm_n}\|}} \right] \leqslant \tau_1$.

Suppose that, if possible

$$\frac{1}{e\rho} \limsup_{\|m\| \to \infty} \left[\|\lambda_{nm_n}\| \{|a_m| \phi_D(m)\}^{\frac{\rho}{\|\lambda_{nm_n}\|}} \right] = \tau_1,\ \eta_1 < \tau_1,\ 0 < \eta_1 < 1.$$

Choosing $\epsilon > 0$ s. t. $\tau_1 \eta_1 + \epsilon = \tau_1' < \tau_1$ we get from Lemma 2, $\log M_{f,D}(r) < \tau_1' e^{r\rho}$, for $r \geqslant r_0$ which implies $\tau_1 \in$ int $T_f(\rho)$, a contradiction.

Proof of the sufficient condition : For any $\epsilon > 0$, it follows from (4.7) that $\exists\ N_0(\epsilon)$ s.t.

$$|a_m| \phi_D(m) < \left(\frac{e\rho(\tau_1+\epsilon)}{\|\lambda_{nm_n}\|} \right)^{\frac{\|\lambda_{nm_n}\|}{\rho}}, \text{ for } \|m\| > N_0 \quad ..(4.8)$$

Arguing similarly as in the Theorem 4, we obtain

$$\limsup_{\|m\| \to \infty} \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{-\log\{|a_m| \phi_D(m)\}} \leqslant \rho.$$

Again from (4.7) $\exists$ a sequence $\{\|m^{(p)}\|\}$ with $\|m^{(p)}\| \to \infty$ as $p \to \infty$ s.t.

$$| a_m | \phi_D (m) > \left(\frac{e\rho(\tau_1 - \epsilon)}{\| \lambda_{n m_n} \|} \right)^{\frac{\| \lambda_{nm_n} \|}{\rho}}, \text{ for } m_j = m_j^{(p)}, j = 1, 2, \ldots n, \quad \ldots(4.9)$$

So, $$\limsup_{\| m \| \to \infty} \frac{\| \lambda_{nm_n} \| \log \| \lambda_{nm_n} \|}{- \log \{ | a_m | \phi_D (m) \}} \geqslant \rho.$$

Thus ρ is the order of f.

Now taking $\tau_1' = \tau_1 + \epsilon$, we find that $\tau_1' > \tau_1$. Then from (4.8)

$$| a_m | \phi_D (m) < \left(\frac{e \rho \tau_1'}{\| \lambda_{n m_n} \|} \right)^{\frac{\| \lambda_{nm_n} \|}{\rho}}, \text{ for } \| m \| > N_0$$

From Lemma 2 and the definition of the type of f, it follows that

$$\tau_1' \in T_f (\rho) . \quad \ldots(4.10)$$

Again by considering $\tau_1'' = \tau_1 - \epsilon$, we observe that $\tau_1'' < \tau_1$. Then from (4.9)

$$| a_m | \phi_D (m) > \left(\frac{e \rho \tau_1''}{\| \lambda_{n m_n} \|} \right)^{\frac{\| \lambda_{nm_n} \|}{\rho}} \text{ for } m_j = m_j^{(p)} ; j = 1, 2, \ldots n$$

Also from $M_{f,D} (r) \geqslant | a_m | \phi_D (m) \exp \| r \lambda_{nm_n} \|$, we have

$$\log M_{f,D} (r) > \frac{\| \lambda_{nm_n^{(p)}} \|}{\rho} \log \frac{e \rho \tau_1''}{\| \lambda_{nm_n^{(p)}} \|} + r \| \lambda_{nm_n^{(p)}} \| , \forall r$$

considering $r = r_p = \frac{1}{\rho} \log \frac{\| \lambda_{nm_n^{(p)}} \|}{\rho \tau_1''}$ in the above inequality, we get

$$\log M_{f,D} (r) > \frac{\| \lambda_{nm_n^{(p)}} \|}{\rho} = \tau_1'' e^{r\rho}, \text{ for } r = r_p > r_0$$

which implies that $\tau_1'' \notin T_f (\rho)$. $\quad \ldots(4.11)$

From (4.10) and (4.11), we obtain that $\tau_1 = \tau (D)$ is the type of f.

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