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SUBSET SELECTION OF BETTER THAN CONTROL CONSTANT FAILURE RATE UNITS USING DIFFERENT SAMPLING SCHEMES

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Subset selection procedure for selecting process better than control out of k constant failure rate processes is considered. The subset selection rules based on three types of sampling schemes, viz. (a) observations are taken till r th failure from each process (SSI); (b) observations from the process are taken till r th failure from control process (SSII); and (c) observations from all processes are taken till a fixed time instant T, (SSIII) are defined. Their performance is compared on the basis of (i) the probability that all processes worse than control are selected in the subset, (ii) the expected proportion of processes better than control included in the selected subset and (*iii*) the expected proportion of processes worse than control included in the selected subset.

1. INTRODUCTION

In many real life situations one is often interested in comparing the new competing brands with a conventional brand in the market in order to discard the new brands that are inferior than the conventional. In reliability discussions suppose there are k manufacturers producing units of constant failure rates $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively, then the units with failure rates $\lambda_i \geqslant \lambda_0$, $i = 1, 2, \ldots k$, is of interest, where λ_0 is the failure rate of the unit already in use.

The statistical decision problem involved here is the following. Let $\pi_1, \pi_2, \ldots, \pi_k$ be k different populations having exponential distributions with parameters λ_1, $\lambda_2, \ldots, \lambda_k$ respectively. Let π_0 be standard or control exponential population with parameter λ_0. Our goal is to select a subset of all those populations from given k populations such that $\lambda_i \geqslant \lambda_0$. Any population falling in this subset is referred to better than control while that falling outside this is called worse than control. Let I denote the set of subscripts from amongst $1, 2, \ldots, k$ of all population better than control and let k_1 be number of these subscripts. The integer k_1 is unknown. Then

$$\Omega = \{\lambda_i \geqslant \lambda_0 \, , \, i \in I, \, \lambda_j < \lambda_0, \, j \notin I\}$$

is the complete parametric space.

Such types of goals have been considered by Paulson (1952), Dunnett (1955), Gupta and Sobel (1958), Krishnaiah and Rizvi (1966) and Bechhofer and Turnbull (1978) in ranking and selection problems. For our problem several sampling schemes can be adopted for obtaining data on test unit from any population. The basic aim of this paper is to define subset selection procedures for the goal mentioned above, based on three different sampling schemes given below and compare the performance of these sampling schemes on the basis of certain criteria of goodness to be defined.

Sampling Scheme I (SSI)—One unit from each of the $(k+1)$ populations π_i, $i = 0, 1, 2, \ldots, k$ is put into operation and as soon as this unit fails, it is replaced by an identical unit instantaneously. Each process is terminated separately at the instant of its r_1th failure.

Sampling Scheme II (SSII)—The sampling procedure is same as in SSI with the difference that sampling is terminated as soon as r_2th failures occur from control population π_0.

Sampling Scheme III (SSIII)—Sampling from $(k+1)$ populations starts and continues in the same manner as in SSI but now sampling is terminated for all processes simultaneously at a fixed time instant T.

Let R_I, R_{II} and R_{III} be subset selection rules based on sampling schemes SSI, SSII and SSIII respectively. We shall compare three selection rules by keeping the total time of test to be the same in the sampling schemes, on the basis of following criteria :

(1) P, the probability that all populations worse than control are selected in subset by the selection rule;

(2) S, the expected proportion of populations better than control included in the selected subset by the selection rule;

(3) J, the expected proportion of populations worse than control included in the selected subset by the selection rule.

It is desirable for a good selection procedure to have $\sup_{\Omega} P$ and $\sup_{\Omega} J$ as small and $\inf_{\Omega} S$ as large as possible. The last two criteria were also considered by Krishnaiah and Rizvi (1966) for their selection problem in multivariate normal populations. As is customary by correct selection (CS) we mean the inclusion of the populations corresponding to the subscripts belonging to I in the selected subset. For any rule R the selection constant is to be chosen such that

$$P\,(CS/R) \geqslant P^* \qquad \ldots(1)$$

for all parametric configurations in Ω, where $P^* \left(\frac{1}{k} \leqslant P^* \leqslant 1 \right)$ a pre-assigned number.

The following functions are frequently used in the paper

$$F_k\,(H, \rho) = \int_{-\infty}^{\infty} \Phi^k \left[\frac{\rho^{1/2} z + H}{(1-\rho)^{1/2}} \right] d\Phi\,(z)$$

and

$$I_{u,v}(\alpha;t) = \int_0^\infty G_u^t\,(x\alpha)\,dG_v\,(x)$$

where Φ is the distribution function of standard normal distribution and G_j is the distribution function of Gamma distribution with parameters $(1, j)$. The functions have been extensively tabulated by Gupta (1963a) and (1963b) respectively.

Selection rules R_I, R_{II} and R_{III} are defined and the exact and asymptotic expression of probability of correct selection, P, S and J are obtained for these rules respectively in section 2, 3 and 4. Lastly, in section 5 we present the comparison of three selection rules on the basis of criteria defined above.

2 Subset Selection Procedure Using SSI

Let us denote by $t_i\,(j)$, the time instant of the jth failure for the ith population, $i = 0, 1, 2, \ldots, k$, $j \geqslant 1$. We base our subset selection rule defined below on statistics $t_i\,(j)$,

Rule R_I : Include π_i in the selected subset if and only if

$$t_i\,(r_1) \leqslant d\,t_0\,(r_1),\quad i = 1, 2, \ldots, k$$

where d is a selection constant with $d \geqslant 1$.

In case control parameter λ_0 is known, the selection rule R_I is modified to R_I' by replacing $t_0\,(r_1)$ by r_1/λ_0. The probability of correct selection for rule R_I and R_I' can be derived by using arguments of Gupta and Sobel (1958). In fact

$$\operatorname*{Inf}_{\Omega} P\,(CS/R_I) = I_{r_1, r_1}(d\,;k_1) \geqslant I_{r_1, r_1}(d\,;k) \qquad \ldots(2)$$

$$\operatorname*{Inf}_{\Omega} P(CS/R_I') = G_{r_1}^{k_1}\,(dr_1) \geqslant G_{r_1}^{k}\,(dr_1). \qquad \ldots(3)$$

The infimum in both cases (2) and (3) is attained when $\lambda_i \to \lambda_0$, $i \in I$.

A normal approximation for the probability of correct selection yields the following asymptotic results for large r_1 corresponding to (2) and (3) respectively.

$$\operatorname*{Inf}_{\Omega} P_a\,(CS/R_I) \geqslant F_k\,(H_1, \rho_1)$$

where

$$H_1 = \frac{\sqrt{r_1}\,(d-1)}{\sqrt{1+d^2}},\quad \rho_1 = \frac{d^2}{1+d^2}$$

and

$$\operatorname*{Inf}_{\Omega} P_a\,(CS/R_I') \geqslant \Phi^k\,[(d-1)\sqrt{r_1}].$$

Denote by $P(I)$, $S(I)$ and $J(I)$ the values of P, S and J for rule R_I. In the following Table I we present the exact and the asymptotic values of $\sup_{\Omega} P(I)$, $\inf_{\Omega} S(I)$ and $\sup_{\Omega} J(I)$ along with the configuration where the infimum or supremum is attained. These expression can be obtained by following arguments similar to that of Krishnaiah and Rizvi (1966).

TABLE 1

	Exact	Asymptotic	Configuration*
$\sup_{\Omega} P(I)$	$I_{r_1, r_1}(d\,;k-k_1)$	$F_{k-k_1}(H_1, \rho_1)$	$\lambda_i \to \lambda_0, i \in \bar{I}$
$\inf_{\Omega} S(I)$	$I_{r_1, r_1}(d\,;1)$	$F_1(H_1, \rho_1)$	$\lambda_i \to \lambda_0, i \in I$
$\sup_{\Omega} J(I)$	$I_{r_1, r_1}(d\,;1)$	$F_1(H_1, \rho_1)$	$\lambda_i \to \lambda_0, i \in \bar{I}$

*where $\bar{I}$ is the complement of I.

3. SUBSET SELECTION PROCEDURE USING SSII

In this sampling scheme (SSII) our statistic is again $t_i(j)$, $i = 0, 1, 2, \ldots, k$, $j \geqslant 1$. The selection rule R_{II} now is given by

Rule R_{II}:—Include π_i in the selected subset iff

$$t_i(r_2 - b) \leqslant t_0(r_2), \; i = 1, 2, \ldots, k$$

where b $(0 \leqslant b \leqslant r_2)$ is a selection constant.

In case when control is known then rule R_{II} is modified to R'_{II} by replacing $t_0(r_2)$ by r_2/λ_0. The probability of correct selection for rule R_{II} when control is unknown, is given by

$$P(CS/R_{II}) = P[t_i(r_2 - b) \leqslant t_0(r_2), \; i \in I].$$

Again the probability of correct selection is minimized as $\lambda_i \to \lambda_0$, $i \in I$ with equality only in limits. Hence using standard arguments as for R_I

$$\inf_{\Omega} P(CS/R_{II}) = I_{r_2 - b, r_2}(1\,;k_1) \geqslant I_{r_2 - b, r_2}(1\,;k). \qquad \ldots(4)$$

For known control the infimum of probability of correct selection is obtained as $\lambda_i \to \lambda_0$, $i \in I$, which is given by

$$\inf_{\Omega} P(CS/R'_{II}) = G^{k_1}_{r_{2-b}}(z) \geqslant G^{k}_{r_{2-b}}(z). \qquad \ldots(5)$$

For large r_2, asymptotically using normal approximation we have

$$\inf_{\Omega} P_a(CS/R_{II}) \geqslant F_k(H_2, \rho_2)$$

where

$$H_2 = \frac{b}{\sqrt{(2r_2 - b)}}, \quad \rho_2 = \frac{r_2}{2r_2 - b}$$

and

$$\inf_{\Omega} P_{\omega} (CS/R'_{II}) \geqslant \Phi^k \left[\frac{b}{\sqrt{(r_2 - b)}} \right]$$

corresponding to (4) and (5) respectively.

Now, letting P(II), S (II) and J(II) denote the values of P, S and J for rule R_{II} and using argument as remarked earlier for rule R_I we have the following Table II.

TABLE II

	Exact	Asymptotic	Configuration
$\sup_{\Omega} P(II)$	$I_{\frac{r}{2}-b, \frac{r}{2}} (1 ; k - k_1)$	$F_{k-k_1} (H_2 , \rho_2)$	$\lambda_i \to \lambda_0 , i \in \bar{I}$
$\inf_{\Omega} S(II)$	$I_{\frac{r}{2}-b, \frac{r}{2}} (1 ; 1)$	$F_1 (H_2 , \rho_2)$	$\lambda_i \to \lambda_0 , i \in I$
$\sup_{\Omega} J(II)$	$I_{\frac{r}{2}-b, \frac{r}{2}} (1 ; 1)$	$F_1 (H_2 , \rho_2)$	$\lambda_i \to \lambda_0 , i \in \bar{I}$

4. Subset Selection Procedure Using SSIII

When one obtains data according to SSIII, it is known that $d_i (T)$, $i = 0, 1, 2, \ldots, k$, the total number of failures upto instant T for ith population, is a sufficient statistics for the parameter λ_i , $i = 0, 1, 2, \ldots, k$.

We base our subset selection rule defined below on $d_i (T)$.

Rule R_{III}:—Include π_i in the selected subset if and only if

$$d_i (T) + 1 \geqslant c d_0 (T), \; i = 1, 2, \ldots, k.$$

where c is a constant with $0 < c \leqslant 1$ and is to be chosen such that condition (1) is satisfied for the rule R_{III}. Again when control is known the selection rule R_{III} is modified to R'_{III} by replacing $d_0 (T)$ by $\lambda_0 T$.

It may be noticed that the selection rule R_{III} is a modification of the conventional rule in that number one has been added in the left hand side. This modification is necessary in order that P^* requirement (1) be satisfied by the selection rule R_{III} for all values of P^*. It can be established easily that for the conventional type selection rule P^* requirement does not hold for large P^*, the values that are of practical interest. This modification is motivated by a result of Chapman (1952) that $X_1/X_2 + 1$ is an unbiased estimator of λ_1/λ_2 in Poisson population.

The probability of correct selection for rule R_{III} when control is unknown, is

$$P (CS/R_{III}) = P [d_i (T) + 1 \geqslant c \, d_0 (T), i \in I]$$

where $d_i (T)$ has Poisson distribution with parameter $\Lambda_i = \lambda_i T$, $i = 0, 1, \ldots, k$.

The probability of correct selection decreases as $\lambda_i \to \lambda_0$, $i \in I$ with equality only in limits. Hence by our usual argument

$$\operatorname*{Inf}_{a} P\,(CS/R_{III}) \geqslant \operatorname*{Inf}_{\lambda_0} \sum_{x=0}^{\infty} \left[\sum_{r=[cx-1]}^{\infty} \frac{e^{-\Lambda_0} \Lambda_0^r}{r!} \right]^k \frac{e^{-\Lambda_0} \Lambda_0^x}{x!} \qquad \ldots(6)$$

where $[y]$ is greatest integer less than or equal to y.

When control is known, the infimum of probability of correct selection is given by when $\lambda_i \to \lambda_0$, $i \in I$. Thus

$$\operatorname*{Inf}_{a} \; P\,(CS/R'_{III}) \geqslant \left[\sum_{r=[c\Lambda_0-1]}^{\infty} \frac{e^{-\Lambda_0} \Lambda_0^r}{r!} \right]^k.$$

For large values of T we can develop an asymptotic expression using normal approximation with a continuity correction of $\frac{1}{2}$ of probability of correct selection for rule R_{III} can be obtained and corresponding to (6) we have

$$\operatorname*{Inf}_{a} P_a\,(CS/R_{III}) \geqslant \operatorname*{Inf}_{\lambda_0} \int_{-\infty}^{\infty} \Phi^k \left(cz + \frac{2\,(1-c)\,\Lambda_0 + 3}{2\sqrt{\Lambda_0}} \right) d\Phi\,(z). \qquad \ldots(7)$$

Infimum of expression on extreme right of (7) is attained at

$$\lambda_0 = \frac{3}{2\,(1-c)\,T} .$$

Hence

$$\operatorname*{Inf}_{a} P_a\,(CS/R_{III}) \geqslant F_k\,(H_3\,,\,\rho_3) \qquad \ldots(8)$$

where

$$H_3 = \frac{c^2}{c^2+1}\,, \quad \rho_3 = \frac{\sqrt{6}\;\sqrt{(1-c)}}{\sqrt{1+c^2}} .$$

Similarly an asymptotic expression when control is known

$$\operatorname*{Inf}_{a} P_a\,(CS/R'_{III}) \geqslant \Phi^k \left[\frac{\Lambda_0\,(1-c)+1)}{\sqrt{\Lambda_0}} \right].$$

Now using the same steps as for R_I and R_{II} we can write an exact and asymptotic expression of P, S and J for the rule R_{III} and we shall get

$$\operatorname*{Sup}_{a} P\,(III) = \sum_{x=0}^{\infty} \left[\sum_{r=0\,[cx-1]}^{\infty} \frac{e^{-\Lambda_0} \Lambda_0^r}{r!} \right]^{k-k_1} \frac{e^{-\Lambda_0} \Lambda_0^x}{x!} \qquad \ldots(9)$$

and

$$\operatorname*{Sup}_{a} P_a\,(III) = F_{k-k_1}\,(H_4\,,\,\rho_4)$$

where

$$H_4 = \frac{2\Lambda_0\,(1-c)+3}{2\sqrt{\Lambda_0}\;\sqrt{1+c^2}}\,, \quad \rho_4 = \frac{c^2}{c^2+1} .$$

TABLE III

Table for comparison among R_I, R_{II} and R_{III} on the basis of P, S, J

(a)

$k = 2, \ k_1 = 1$

P^*	Criterion	Λ \ Rule	R_I	R_{II}	R_{III}
.90	$p = s = j$	10	0.97128	0.9452	0.9861
		16	0.97725	0.9482	0.99374
.95	$p = s = j$	10	0.98214	0.97128	0.9953
		16	0.98928	0.97128	0.99903
.99	$p = s = j$	10	0.99653	0.99653	0.99931
		16	0.99744	0.99653	0.99981

(b)

$k = 3$

k_1	Criterion	Λ \ P^* Rule	.90 R_I	.90 R_{II}	.90 R_{III}	.95 R_I	.95 R_{II}	.95 R_{III}
1	p	10	0.96786	0.95382	0.9788	0.9844	0.98264	0.99628
		16	0.97169	0.9545	0.9931	0.98756	0.98313	0.99953
2	$p = s^* = j^*$	10	0.97725	0.97128	0.98928	0.9937	0.98928	0.99813
		16	0.98214	0.97128	0.99653	0.9937	0.9918	0.99977

(c)

$k = 4$

k_1	Criterion	Λ \ P^* Rule	.90 R_I	.90 R_{II}	.90 R_{III}	.95 R_I	.95 R_{II}	.95 R_{III}
1	p	10	0.96946	0.94092	0.98161	0.98533	0.97765	0.9971
		16	0.97151	0.94798	0.99596	0.9909	0.9816	0.9993
2	p	10	0.97451	0.95382	0.98766	0.98796	0.9826	0.99807
		16	0.97772	0.95954	0.9973	0.9927	0.9860	0.99953
3	$p = s^* = J^*$	10	0.98214	0.97128	0.99374	0.9948	0.9892	0.99903
		16	0.9861	0.97128	0.99865	0.9953	0.9918	0.99977

* These values of s or j also apply to all other valuesof k_1 in that table.

$$\inf_{\Omega} S\,(III) = \sup_{\Omega} J\,(III) = \sum_{x=0}^{\infty} \Big[\sum_{r=[cx-1]}^{\infty} \frac{e^{-\Lambda_0}\,\Lambda_0^r}{r!} \Big] \times \frac{e^{-\Lambda_0}\,\Lambda_0^x}{x!}$$

and

$$\inf_{\Omega} S_a\,(III) = \sup_{\Omega} J_a\,(III) = F_1\,(H_4\,,\,\rho_4)\,. \qquad \ldots(10)$$

5. Comparison and Conclusions

The comparison of three selection rules will be done under the equality configuration $\lambda_i = \lambda_0$, $i = 1, 2, \ldots, k$. For comparing the three selection rules on the basis of P, S and J for various given values of λ_0, first of all we equate the expected total time of test for the three procedures under the equality configuration, getting

$$\frac{r_1\,(k+1)}{\lambda_0} = \frac{r_2\,(k+1)}{\lambda_0} = (k+1)\,T$$

for given λ_0 and T, we must have $r_1 = r_2 = \Lambda$ where $\Lambda = \lambda_0 T$. For implementation of rules R_I and R_{II} we choose r_1, r_2 as the nearest integer to Λ.

Then for given k and P^* and r_1, r_2, T as fixed above we calculate the value of the selection constants d, b and c such that condition (1) is satisfied by the rule R_I, R_{II} and R_{III} respectively, i.e., d, b, c are obtained so that right hand side of (2), (4) and (8) equal to P^*. The values of d, b, c so obtained are used in computing the values of $p = \sup_{\Omega} P$, $s = \inf_{\Omega} S$ and $j = \sup_{\Omega} J$ from Table I for rule R_I, Table II for rule R_{II} and (9) and (10) for rule R_{III} respectively.

Table III shows the comparison between three rules on the basis of asymptotic values of p, s and j for given values of k, P^* and Λ. While presenting the tables we have used the fact that for any of the three rules for a given k the values of s and j do not depend on the value of k_1, and $s = j$. Further if $k_1 = k - 1$ then $p = s = j$. From the tables we conclude that the selection rule R_{II} based SSII is best with respect to criteria P and J while the rule R_{III} based on SSII is best with respect to criterion S. The selection rule R_I based on SSI has second rank with respect to all the criteria P, S and J.

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