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ON THE GROWTH OF GENERALIZED AXISYMMETRIC POTENTIALS

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In the present paper we have introduced some growth parameters for the solutions H of the partial differential equation $\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+\frac{2\alpha\partial H}{y\partial y}=0$, $\alpha>0$, which are regular in the disc $D_R=\{(x,y): x^2+y^2<R^2\}$, $0<R<\infty$. Characterizations of these growth parameters have been obtained in terms of the coefficients in the ultra-spherical harmonic expansion of H.

1. INTRODUCTION

In this paper we consider an elliptic partial differential equation

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+\frac{2\alpha}{y}\frac{\partial H}{\partial y}=0, \qquad \alpha>0 \qquad \ldots(1.1)$$

and, for the solutions of (1.1), develop results analogous to those found in the theory of analytic functions of one complex variable. The later theory is recently developed, and, being essentially equivalent to the study of Laplace's equation in two dimensions suggests the possibility of fruitful extensions to more general elliptic equations.

The solutions H of (1.1), regular about origin, can be expanded as

$$H(r,\theta)=\sum_{n=0}^{\infty} a_n r^n C_n^{\alpha}(\cos\theta) \qquad \ldots(1.2)$$

and are called generalized axisymmetric potentials (GASP's) (Weinstein 1953). Here $x=r\cos\theta$, $y=r\sin\theta$ and $C_n^{\alpha}(x)$ are Gegenbauer polynomials. The series on the right hand side of (1.2) converges uniformly in $0\leqslant r\leqslant r_0$ for some $r_0>0$.

Let $D_R=\{(x,y): x^2+y^2<R^2\}$, $0<R\leqslant\infty$. Denote by H_R the class of all GASP's H, such that the series on the right hand side of (1.2) converges uniformly on compact subsets of D_r' for every $r'\leqslant R$ but for no $r'>R$. Clearly, H_∞ consists of all entire GASP's.

Set $M(r,H)=\max_{0\leqslant\theta\leqslant 2\pi}|H(r,\theta)|$. Then, the order $\rho_\infty(H)$ of an entire GASP H is defined as $\rho_\infty(H)=\limsup_{r\to\infty}\log\log M(r,H)/\log r$, and if $0<\rho_\infty(H)<\infty$ the type $T_\infty(H)$ of H is defined as $T_\infty(H)=\limsup_{r\to\infty}\log M(r,H)/r^{\rho_\infty(H)}$. Coefficient characterization of $\rho_\infty(H)$ was obtained by Fryant (1977). Coefficient characterization of $T_\infty(H)$ was found by Gilbert (1969, Theorem 4. 3. 4).

However, the results of Fryant and Gilbert are not applicable to the GASP's $H \in H_R$, $0 < R < \infty$. In the present paper we study the growth of such GASP's. Thus, we first introduce the concepts of order and type of a GASP $H \in H_R$.

A GASP $H \in H_R$, $0 < R < \infty$, will be said to be of order ρ if

$$\rho = \limsup_{r \to R} \frac{\log^+ \log^+ M(r, H)}{\log (R/(R-r))}$$

where $\log^+ x = \max(0, \log x)$. If $0 < \rho < \infty$, then the type T of H is defined as

$$T = \limsup_{r \to R} \frac{\log^+ M(r, H)}{(R/(R-r))^\rho}. \qquad \ldots(1.3)$$

In Section 2, we obtain coefficient characterizations of order ρ and type T of $H \in H_R$.

2. Coefficient Characterizations

We first need the definitions of order and type of a function of a complex variable analytic in the disc $|z| < R$, $0 < R < \infty$.

Let $f(z)$ be analytic in $|z| < R$. Then the order ρ_0 of $f(z)$ is defined as

$$\rho_0 = \lim \underset{r \to R}{\text{psu}} \frac{\log^+ \log^+ m(r, f)}{\log (R/(R-r))}$$

$$m(r, f) = \max_{|z| = r} |f(z)|$$

and if $0 < \rho_0 < \infty$, then the type T_0 of $f(z)$ is defined as

$$T_0 = \limsup_{r \to R} \frac{\log^+ m(r, f)}{(R/(R-r))^{\rho_0}}.$$

We now have the following lemmas.

Lemma 1 (Beuermann 1931; Maclane 1963, p. 47) — Let $f(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in $|z| < R$ $(0 < R < \infty)$ and has order ρ_0 $(0 \leqslant \rho_0 \leqslant \infty)$. Then,

$$\rho_0 = \limsup_{n \to \infty} (\log^+ \log^+ |b_n| R^n)/(\log n - \log^+ \log^+ |b_n| R^n).$$

Lemma 2 (Bajpai *et al.* 1974; Kapoor 1972, p. 156) — Let $f(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in $|z| < R$ $(0 < R < \infty)$. Then $f(z)$ is of order ρ_0 $(0 < \rho_0 < \infty)$ and type T_0, if and only if $\nu_0 = (\rho_0 + 1)^{\rho_0 + 1} T_0 / \rho_0^{\rho_0}$, where

$$\nu_0 = \limsup_{n \to \infty} (\log^+ |b_n| R^n)^{\rho_0 + 1}/n^{\rho_0}$$

satisfies $0 < \nu_0 < \infty$.

Lemma 3 (Gilbert 1969, Theorem 4. 2. 7) — A GASP $H \in H_R$ $(0 < R \leqslant \infty)$, if and only if

$$\limsup_{n \to \infty} |a_n|^{1/n} = 1/R.$$

Lemma 4 — Let $H \in H_R$ $(R > 0)$. Then, for $0 < r < R$, we have

$$r^n \mid a_n \mid \leqslant M(r, H) K(\alpha) \left(\frac{(n+\alpha)\Gamma(n+1)}{\Gamma(n+2\alpha)}\right)^{1/2} \qquad \ldots(2.1)$$

where

$K(\alpha) = ((2\pi)^{1/2}\Gamma(2\alpha))/(2^{\alpha}\Gamma(\alpha+1/2))$.

PROOF: It is known (Szegö 1967, p. 82) that

$$\int_{-1}^{1} (1 - x^2)^{\alpha-1/2} C_n^{\alpha}(x) C_m^{\alpha}(x)\, dx$$

$$= \frac{\delta_{nm} 2^{2\alpha-1}\Gamma(n+2\alpha)}{(n+\alpha)\Gamma(n+1)} \left(\frac{\Gamma(\alpha+1/2)}{\Gamma(2\alpha)}\right)^2 \qquad \ldots(2.2)$$

where $\delta_{nm} = 1$ if $m=n$ and $\delta_{nm} = 0$ otherwise. Now, using the uniform convergence of H on compact subsets of D_R and (2.2), we get for $r < R$

$$\frac{a_n r^n 2^{2\alpha-1}\Gamma(n+2\alpha)}{(n+\alpha)\Gamma(n+1)} \left(\frac{\Gamma(\alpha+1/2)}{\Gamma(2\alpha)}\right)^2 = \int_0^{\pi} \sin^{2\alpha}\theta\, C_n^{\alpha}(\cos\theta) H(r, \theta)\, d\theta.$$

Thus, using Schwartz's inequality and (2.2) we get

$$\frac{\mid a_n \mid r^n 2^{2\alpha-1}\Gamma(n+2\alpha)}{(n+\alpha)\Gamma(n+1)} \left(\frac{\Gamma(\alpha+1/2)}{\Gamma(2\alpha)}\right)^2 \leqslant M(r, H) \int_0^{\pi} \sin^{2\alpha}\theta \mid C_n^{\alpha}(\cos\theta) \mid d\theta$$

$$\leqslant M(r, H) \left(\int_0^{\pi} \sin^{2\alpha}\theta \mid C_n^{\alpha}(\cos\theta) \mid^2 d\theta\right)^{1/2} \left(\int_0^{\pi} \sin^{2\alpha}\theta\, d\theta\right)^{1/2}$$

$$\leqslant M(r, H) \left[\frac{\pi\, 2^{2\alpha-1}\Gamma(n+2\alpha)}{(n+\alpha)\Gamma(n+1)} \left(\frac{\Gamma(\alpha+1/2)}{\Gamma(2\alpha)}\right)^2\right]^{1/2}.$$

Now, (2.1) follows from the above relation. This proves the lemma.

We now prove:

Theorem 1 — Let GASP $H \in H_R$ $(0 < R < \infty)$, given by (1.2), be of order ρ. Then

$$\rho = \limsup_{n\to\infty} \frac{\log^+ \log^+ \mid a_n \mid R^n}{\log n - \log^+ \log^+ \mid a_n \mid R^n}. \qquad \ldots(2.3)$$

PROOF: Let the limit superior on the right hand side of (2.3) be denoted by d. Clearly $0 \leqslant d \leqslant \infty$. First, let $0 < d < \infty$. Let d' be an arbitrary number such that $0 < d' < d$. Then, by the definition of d there exists a sequence $\{n_k\}$ of positive integers tending to ∞ such that

$$\log \mid a_{n_k} \mid R^{n_k} > n_k^{d'/(1+d')} \qquad \ldots(2.4)$$

for $k = 1, 2, 3, \ldots$ Using (2.4) and Lemma 4, we get

$$\log M(r, H) > n_k^{d'/(1+d')} + n_k \log (r/R) - \log K(\alpha)$$

$$-\tfrac{1}{2} \log \left(\frac{(n_k+\alpha)\, \Gamma(n_k+1)}{\Gamma(n_k+2\alpha)} \right) \qquad \ldots(2.5)$$

for the sequence $\{n_k\}$ and all $r < R$. Let $\{r_k\}$ be the sequence defined by $n_k = \{((1+d')/d') \log (R/r_k)\}^{-(1+d')}$, $k = 1, 2, 3, \ldots$ Then $r_k \to R$ as $k \to \infty$. Now, for the sequence $\{r_k\}$, (2.5) gives

$$\log M(r_k, H) > \frac{d'^{d'}}{(1+d')^{1+d'}} (\log (R/r_k))^{-d'} - \log K(\alpha)$$

$$+ (1-\alpha)(1+d') \log \{((1+d')/d') \log (R/r_k)\} + o(1),$$

since $\Gamma(n+2\alpha)/\Gamma(n) \sim n^{2\alpha}$ as $n \to \infty$ ([Rudin 1976, p. 193]). The above inequality gives that

$$\limsup_{k \to \infty} \frac{\log^+ \log^+ M(r_k, H)}{\log (R/(R-r_k))} \geqslant d'. \qquad \ldots(2.6)$$

Since d' $(< d)$ is arbitrary, (2.6) gives that

$$\rho \geqslant d. \qquad \ldots(2.7)$$

Obviously, (2.7) holds for $d = 0$. For $d = \infty$, the above arguments give that $\rho = \infty$.

On the other hand, it is known (Szegö 1967, p. 97) that,

$$r^n C_n^\alpha (\cos \theta) = \frac{2^{1-2\alpha}\, \Gamma(n+2\alpha)}{\Gamma(n+1)\, (\Gamma(\alpha))^2} \int_0^\pi (x + i\, y \cos \varphi)^n \sin^{2\alpha-1} \varphi \, d\varphi$$

where $x = r \cos \theta$ and $y = r \sin \theta$. Thus, for $0 \leqslant \theta \leqslant 2\pi$, we get

$$r^n \mid C_n^\alpha (\cos \theta) \mid \leqslant \frac{r^n\, 2^{1-2\alpha}\, \Gamma(n+2\alpha)}{\Gamma(n+1)\, (\Gamma(\alpha))^2} \int_0^\pi \sin^{2\alpha-1} \varphi \, d\varphi$$

$$= r^n \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\, \Gamma(2\alpha)}. \qquad \ldots(2.8)$$

Using (1.2) and (2.8) we get

$$\mid H(r, \theta) \mid \leqslant \frac{1}{\Gamma(2\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} \mid a_n \mid r^n$$

for $0 \leqslant \theta \leqslant 2\pi$ and so

$$M(r, H) \leqslant m(r, g)/\Gamma(2\alpha) \qquad \ldots(2.9)$$

where $g(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} \mid a_n \mid z^n$. Since $\Gamma(n+2\alpha)/\Gamma(n+1) \sim n^{2\alpha-1}$ as $n \to \infty$, it follows, from Lemma 3, that $g(z)$ is analytic in $|z| < R$. Using (2.9) and Lemma 1 for $g(z)$ we get

$$\rho \leqslant d. \qquad \ldots(2.10)$$

Combining (2.7) and (2.10) we get (2.3). This proves the theorem.

Theorem 2 — Let GASP $H \in H_R$ $(0 < R < \infty)$ be given by (1.2). Then, H is of order ρ $(0 < \rho < \infty)$ and type T, if and only if

$$\nu = \frac{(\rho+1)^{\rho+1}}{\rho^{\rho}} T$$

where

$$\nu = \limsup_{n \to \infty} \frac{(\log^{+} | a_n | R^n)^{\rho+1}}{n^{\rho}} \qquad \ldots(2.11)$$

satisfies $0 < \nu < \infty$.

PROOF : First, supose $0 < \nu < \infty$, then (2.11) gives that

$$\rho = \limsup_{n \to \infty} \frac{\log^{+} \log^{+} | a_n | R^n}{\log n - \log^{+} \log^{+} | a_n | R^n}$$

and so, by Theorem 1, H is of order ρ.

Next, let H be of type T, $T < \infty$. Then, for a given $\epsilon > 0$ (1.3) gives that there exists $r^0 = r^0(\epsilon)$ such that

$$\log M(r, H) \leqslant (T+\epsilon)(R/(R-r))^{\rho} \qquad \ldots(2.12)$$

for $r^0 < r < R$. Using (2.12) and Lemma 4, we get

$$\log^{+} | a_n | R^n \leqslant (T+\epsilon)(R/(R-r))^{\rho} + n \log (R/r) + \log^{+} K(\alpha)$$
$$+ \frac{1}{2} \log^{+} \left(\frac{\Gamma(n+1)(n+\alpha)}{\Gamma(n+2\alpha)} \right) \qquad \ldots(2.13)$$

for all n and $r^0 < r < R$. Choose a sequence $\{r'_n\}$ as

$$R/(R - r'_n) = (n/(T+\epsilon))^{\rho 1/(1+\rho)}. \qquad \ldots(2.14)$$

Clearly $r'_n \to R$ as $n \to \infty$. Using (2.13) and (2.14) we obtain

$$\log^{+} | a_n | R^n \leqslant \frac{(T+\epsilon)^{1/(1+\rho)} n^{\rho/(1+\rho)}}{\rho^{\rho/(1+\rho)}} (1+\rho+o(1)).$$

This, on proceeding to the limits, easily gives that

$$\nu \leqslant \frac{(\rho+1)^{\rho+1}}{\rho^{\rho}} T.$$

The reverse inequality is obtained from (2.9) by applying Lemma 2 to the function $g(z)$. Moreover, the converse part of the theorem follows in a straightforward manner from the first part. This proves the theorem.

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