Indian J. pure appl. Math., 13(11) : 1246–1252, November 1982

ON THE GENERALIZED ORDERS OF AN ENTIRE FUNCTION OF SLOW GROWTH

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(*Received* 28 *May* 1981; *after revision* 23 *March* 1982)

The concepts of generalized order, generalized lower order, generalized regular growth and generalized irregular growth are introduced in Kapoor and Nautiyal (1981) to study precisely the growth of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ of slow growth. In the present paper, a necessary and sufficient condition for $f(z)$ to be of generalized regular growth is obtained in terms of a_n's and λ_n's. Further, a decomposition theorem is proved for an entire function of generalized irregular growth.

1. INTRODUCTION

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \qquad \ldots(1.1)$$

be a nonconstant entire function. Here $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers such that no element of the sequence $\{a_n\}_{n=1}^{\infty}$ is zero. Set, $M(r) \equiv M(r,f) = \max_{|z|=r} | f(z) |$; $\mu(r) \equiv \mu(r, f) = \max_{n\geqslant 0} \{ | a_n | r^{\lambda_n}\}$ and $\nu(r) \equiv \nu(r, f) = \max \{\lambda_n : \mu(r) = | a_n | r^{\lambda_n}\}$. Then, $M(r)$, $\mu(r)$ and $\nu(r)$ are called respectively the maximum modulus, the maximum term and the rank of the maximum term of $f(z)$ for $| z | = r$. Elements in the range set of $\nu(r)$ are called principal indices.

Let L° be the class of functions $h(x)$ satisfying the following conditions (H, i) and (H, ii) :

(H, i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$.

(H, ii) $\lim_{x\to\infty} \frac{h(x(1+\tilde{g}(x)))}{h(x)} = 1$ for every function $\tilde{g}(x)$ such that $\tilde{g}(x) \to 0$ as $x \to \infty$.

Let Δ be the class of functions $h(x)$ satisfying the conditions (H, i) and (H, iii) :

(H, iii) $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$ for every c, $0 < c < \infty$.

Following Šeremeta (1970), Shah (1977) defined generalized (α, β)-order $\rho(\alpha, \beta, f)$ and generalized lower (α, β)-order $\lambda(\alpha, \beta, f)$ of an entire function $f(z)$ as

$$\begin{matrix} \rho(\alpha, \beta, f) \\ \lambda(\alpha, \beta, f) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,f))}{\beta(\log r)} \quad \ldots(1.2)$$

where $\alpha(x) \in \Delta$ and $\beta(x) \in L^{\circ}$. Taking $\alpha(x) = \log x$ in (1.2), $\rho(\alpha, \beta, f)$ reduces to the classical order ρ of an entire function. An entire function $f(z)$ for which $\rho = 0$ is said to be of slow growth.

Šeremeta (1970) and Shah (1977), while obtaining coefficient characterizations of generalized (α, β)-order and generalized lower (α, β)-order, have assumed that $\alpha(x) \in \Delta$ and $\beta(x) \in L^{\circ}$ satisfy

$$\frac{d(\beta^{-1}(\alpha(x)))}{d(\log x)} = O(1) \text{ as } x \to \infty \quad \ldots(1.3)$$

besides some other conditions. Clearly (1.3) is not satisfied for the case $\beta(x) = \alpha(x)$. Thus, the growth parameters, given by (1.2), and their coefficient characterizations do not give any precise information about the growth of a class of entire functions of slow growth. For overcoming this difficulty in a precise measurement of the growth of entire functions of slow growth by their Taylor coefficients, in Kapoor and Nautiyal (1981) the generalised orders of an entire function were defined in new way. Thus, let Ω be the class of all functions $h(x)$ satisfying (H, i) and (H, iv) :

(H, iv) there exists a $\delta(x) \in \Delta$ and x_0, k_1 and k_2 such that

$$0 < k_1 \leqslant \frac{d(h(x)))}{d(\delta(\log x))} \leqslant k_2 < \infty.$$

for all $x > x_0$.

Let $\bar{\Omega}$ be the class of all function s $h(x)$ satisfying (H, i) and (H, v) :

$$(H, v) \lim_{x \to \infty} \frac{d(h(x))}{d(\log x)} = k, 0 < k < \infty.$$

It is easily seen that Ω and $\bar{\Omega}$ are contained in Δ. Further Ω and $\bar{\Omega}$ have no common element. The nature of functions in the classes Ω and $\bar{\Omega}$ is illustrated by various examples in Kapoor and Nautiyal (1981). Now, with $\alpha(x)$ either in Ω or in $\bar{\Omega}$, the generalized order $\rho(\alpha, \alpha, f)$ and the generalized lower order $\lambda(\alpha, \alpha, f)$ of an entire function $f(z)$, given by (1.1), are defined as Kapoor and Nautiyal (1981)

$$\begin{matrix} \rho(\alpha, \alpha, f) \\ \lambda(\alpha, \alpha, f) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r, f))}{\alpha(\log r)}. \quad \ldots(1.4)$$

We have $1 \leqslant \lambda(\alpha, \alpha, f) \leqslant \rho(\alpha, \alpha, f) \leqslant \infty$.

An entire function $f(z)$, given by (1.1), for which the generalized order and the generalized lower order, defined in (1.4), are the same is said to be of generalized regular growth. An entire function $f(z)$ which is not of generalized regular growth is said to be of generalized irregular growth.

The coefficient characterizations of the generalized order $\rho(\alpha, \alpha, f)$ and the generalized lower order $\lambda(\alpha, \alpha, f)$, extending some results in Juneja *et al.* (1976), are found in Kapoor and Nautiyal (1981). In the present paper we find a necessary and sufficient condition for an entire function to be of generalized regular growth and obtain a decomposition theorem for an entire function of generalized irregular growth. Our results include some of the results in Juneja *et al.* (1976) and supplement the results of Bajpai (1976).

The following notations will be used in the sequel :

Notation 1—$P(\xi) = \max(1, \xi)$ if $\alpha(x) \in \Omega$,

$$= 1 + \xi \quad \text{if } \alpha(x) \in \bar{\Omega}$$

$\xi \equiv \xi(\alpha)$ being a function of $\alpha(x)$.

Notation 2—$G[x; c] = \alpha^{-1}[c\alpha(x)]$, c is a positive constant.

2. Functions of Generalized Regular Growth

We first prove :

Theorem 1—Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function. Then,

$$\varphi_2 \leqslant \varphi_1 \lim_{n\to\infty} \inf \frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})}$$

where $\varphi_1 \equiv \varphi_2(\alpha)$ and $\varphi_2 \equiv \varphi_2(\alpha)$, for $\alpha(x) \in \Delta$, are defined as

$$\begin{matrix}\varphi_1 \\ \varphi_2\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\nu\ r))}{\alpha\ (\log r)}.$$

Proof : Let $\lim_{n\to\infty} \inf \frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})} = d$. If $d' > d$, we have

$$\alpha(\lambda_{n_t}) < d'\, \alpha(\lambda_{n_t} + 1)$$

for a sequence $\{n_t\}_{t=0}^{\infty}$ such that $n_t \to \infty$. Let r_t be a value of r at which $\nu(r)$ jumps from a value less than or equal to λ_{n_t} to a value greater than or equal to λ_{n_t+1}. Then

$$\alpha(\nu(r_t - 0)) \leqslant \alpha(\lambda_{n_t}) < d'\, \alpha(\lambda_{n_t} + 1)$$
$$\leqslant d'\, \alpha(\nu(r_t + 0))$$

and so, since $\alpha(x) \in \Delta$, we have

$$\varphi_2 \leqslant \lim_{t\to\infty} \sup \frac{\alpha(\nu(r_t - 0))}{\alpha(\log r_t)} \leqslant d' \lim_{t\to\infty} \sup \frac{\alpha(\nu(r_t + 0))}{\alpha(\log r_t)}$$
$$\leqslant d'\, \varphi_1.$$

Since the above inequality is true for arbitary $d' > d$, we get $\varphi_2 \leqslant d\varphi_1$. The theorem is thus proved.

Corollary—1 Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function with generalized orders $\rho(\alpha, \alpha, f)$ and $\lambda(\alpha, \alpha, f)$. Then,

(i) if $\alpha(x) \in \Omega$ and $\lambda(x, \alpha, f) > 1$, we have

$$\lambda(\alpha, \alpha, f) \leqslant \rho(\alpha, \alpha, f) \liminf_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})}.$$

(ii) if $\alpha(x) \in \bar{\Omega}$, we have

$$(\lambda(\alpha, \alpha, f) - 1) \leqslant (\rho(\alpha, \alpha, f) - 1) \liminf_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})}.$$

Corollary 2—Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of generalised regular growth with $1 < \rho(\alpha, \alpha, f) < \infty$. Then

$$\alpha(\lambda_n) \sim \alpha(\lambda_{n+1}) \text{ as } n \to \infty.$$

Corollaries 1 and 2 follow easily from Theorem 3 of Kapoor and Nautiyal (1981) and Theorem 1 and we omit their proofs.

The following Corollary 3 is a refinement of Corollary 2.

Corollary 3 – Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function. Then, a necessary and sufficient condition that $f(z)$ be of generalized regular growth with $1 < \rho(\alpha, \alpha, f) < \infty$ is that for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that for all $n > n_0$, we have

$$|a_n| < \exp\left\{- \lambda_n G\left[\lambda_n ; \frac{1}{\rho^* + \epsilon}\right]\right\} \qquad \text{...(2.1)}$$

where $\rho^* = \rho(\alpha, \alpha, f)$ if $\alpha(x) \in \Omega$ and $\rho^* = \rho(\alpha, \alpha, f) - 1$ if $\alpha(x) \in \bar{\Omega}$, and that there exists a strictly increasing sequence $\{n_p\}_{p=0}^{\infty}$ of positive integers such that

$$\alpha(\lambda_{n_{p+1}}) \sim \alpha(\lambda_{n_p}) \text{ as } p \to \infty \qquad \text{...(2.2)}$$

and

$$P(L_0) = \rho(\alpha, \alpha, f). \qquad \text{...(2.3)}$$

where

$$L_0 \equiv L_0(x) = \lim_{p \to \infty} \frac{\alpha(\lambda_{n_p})}{\alpha\left(\frac{1}{\lambda_{n_p}} \log |a_{n_p}|^{-1}\right)}.$$

PROOF : We first prove the necessary part. Let $f(z)$ be an entire function of generalised regular growth with $1 < \rho(\alpha, \alpha, f) < \infty$. Then the coefficients a_n satisfy (2.1) follows from Theorem 4 of Kapoor and Nautiyal (1981). Now, consider $g(z) = \sum_{p=0}^{\infty} a_{n_p}, z^{\lambda_{n_p}}$, where $\{\lambda_{n_p}\}_{p=0}^{\infty}$ is the sequence of the principal indices of $f(z)$. It is easily seen that $g(z)$ is an entire function and that for any z, $f(z)$ and $g(z)$ have the same maximum term and the rank of the maximum term. Thus, by Theorem 3 of Kapoor and Nautiyal (1981), $g(z)$ is also of generalised regular growth with the generalised order $\rho(\alpha, \alpha, f)$. Since $1 < \rho(\alpha, \alpha, f) < \infty$, applying Corollary 2 to $g(z)$ we get $\alpha(\lambda_{n_p}) \sim \alpha(\lambda_{n_{p+1}})$ as $p \to \infty$. Further, since

$$\psi\,(n_p) \;=\; \frac{1}{\lambda_{n_{p+1}} - \lambda_{n_p}} \log \left| \frac{a_{n_p}}{a_{n_{p+1}}} \right| \text{ is a strictly increasing function of } p, \quad (2.3)$$

follows from Theorems 4 and 5 of Kapoor and Nautiyal (1981).

Sufficiency part follows easily on using Theorem 4 and Lemma 1 of Kapoor and Nautiyal (1981).

3. A Decomposion Theorem

Theorem 2—Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of generalized irregular growth and let $\lambda(\alpha, \alpha, f) < u(\alpha) < \rho(\alpha, \alpha, f)$. Then $f(z)$ is of the form $g_u(z) + h_u(z)$, where $g_u(z)$ is an entire function with generalised order less than or equal to u and $h_u(z) = \sum_{p=0}^{\infty} a_{m_p} z^{\lambda_{m_p}}$ satisfies

$$\lambda(\alpha, \alpha, f) \geqslant u(\alpha) \lim_{p\to\infty} \sup \frac{\alpha(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})}\,.$$

Proof : Denote $u \equiv u(\alpha)$. Let $g_u(z) = \Sigma' a_n z^{\lambda_n}$, where Σ' denotes summation over n for which

$$|a_n| \leqslant \exp\left(-\lambda_n G\left[\lambda_n ; \frac{1}{u^*}\right]\right),$$

$u^* = u$ if $\alpha(x) \in \Omega$ and $u^* = u - 1$ if $\alpha(x) \in \overline{\Omega}$. It follows from Theorem 4 of Kapoor and Nautiyal (1981), that the generalised order $\rho(\alpha, \alpha, g_u)$ of $g_u(z)$ is less than or equal to u.

Now, let $h_u(z) = f(z) - g_u(z) = \sum_{p=0}^{\infty} a_{m_p} z^{\lambda_{m_p}}$. Then

$$|a_{m_p}| > \exp\left(-\lambda_{m_p} G\left[\lambda_{m_p} ; \frac{1}{u^*}\right]\right). \qquad \ldots(3.1)$$

If $\alpha(x) \in \Omega$, we take $r_p = \exp\,(1 + G\,[\lambda_{m_p}\,; 1/u])$. Then for $r_p \leqslant r \leqslant r_{p+1}$, we have by Cauchy's inequality and (3.1) that

$$\log M(r, f) \geqslant \log|a_{m_p}| + \lambda_{m_p} \log r \geqslant \log|a_{m_p}| + \lambda_{m_p} \log r_p > \lambda_{m_p}$$

and so

$$\frac{\alpha\,(\log M\,(r, f))}{\alpha\,(\log r)} > \frac{\alpha\,(\lambda_{m_p})}{\alpha\,(\log r_{p+1})} = \frac{\alpha\,(\lambda_{m_p})}{\alpha\,(1+G)\,[\lambda_{m_{p+1}}\,;\, 1/u]}\,.$$

This, on proceeding to limits and using the fact that $\alpha(x) \in \Omega$, gives

$$\lambda\,(\alpha, \alpha, f) \geqslant u \lim_{p\to\infty} \inf \frac{\alpha\,(\lambda_{m_p})}{\alpha\,(\lambda_{m_{p+1}})}$$

If $\alpha\,(x) \in \overline{\Omega}$, we take $r'_p = \exp\left(\frac{u}{u-1}\, G\left[\lambda_{m_p}\,;\, \frac{1}{u-1}\right]\right)$.

Then, for $r'_p \leqslant r \leqslant r'_{p+1}$, we have by Cauchy's inequality and (3.1) that

$$\log M(r,f) \geqslant \log |a_{m_p}| + \lambda_{m_p} \log r'_p$$
$$> \frac{1}{u-1} \lambda_{m_p} G\left[\lambda_{m_p} ; \frac{1}{u-1}\right]$$

and so

$$\alpha((u-1)\log M(r,f)) > \alpha\left(\lambda_{m_p} G\left[\lambda_{m_p} ; \frac{1}{u-1}\right]\right)$$
$$= \frac{1}{u-1} \alpha(\lambda_{m_p}) + \log(\lambda_{m_p}) \left.\frac{d(\alpha(x))}{d(\log x)}\right|_{x=x^*(\lambda_{m_p})}$$

where $G\left[\lambda_{m_p} ; \frac{1}{u-1}\right] < x^*(\lambda_{m_p}) < \lambda_{m_p} G\left[\lambda_{m_p} ; \frac{1}{u-1}\right]$.

Thus, for $r'_p \leqslant r \leqslant r'_{p+1}$, we have

$$\frac{\alpha((u-1)\log M(r,f))}{\alpha\left(\frac{u-1}{u}\log r\right)} > \frac{\frac{1}{u-1}\alpha(\lambda_{m_p})}{\alpha\left(\frac{u-1}{u}\log r'_{p+1}\right)} + \frac{\log(\lambda_{m_p})}{\alpha\left(\frac{u-1}{u}\log r'_{p+1}\right)} \left.\frac{d(\alpha(x))}{d(\log x)}\right|_{x=x^*(\lambda_{m_p})}$$
$$= \frac{\alpha(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})} + \frac{(u-1)\log(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})} \left.\frac{d(\alpha(x))}{d(\log x)}\right|_{x=x^*(\lambda_{m_p})}$$

Since $\alpha(x) \bar{\Omega}$, on taking limits the above inequality gives

$$\lambda(\alpha, \alpha, f) \geqslant u \liminf_{p\to\infty} \frac{\alpha(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})}.$$

This proves the theorem.

Remarks : (i) Taking $\alpha(x) = \log_j x$, $j = 1, 2, 3, \ldots$, where $\log_1 x = \log x$ and $\log_j x = \log(\log_{j-1} x)$, results of this paper give some results of Juneja *et al.* (1976).

(ii) A result analogous to Theorem 2 for the classical order of an entire function was obtained by Whittaker (1933).

ACKNOWLEDGEMENT

The authors are thankful to the referee for his several helpful comments concerning the contents of the paper.

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