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MOTION INDUCED BY AXISYMMETRIC BODIES VIBRATING ALONG THE AXIS OF ROTATION OF AN INFINITE ROTATING LIQUID

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In the present paper we have investigated the motion of axisymmetric bodies viz. that of a prolate spheroid and a paraboloid of revolution, when each of them is set to vibrate harmonically at time $t = 0$. The investigation of the present paper reveals that whereas in the ultimate motion a singularity arises in the case of the spheroid; no such singularity occurs in the case of the paraboloid. It has been further shown that the results obtained in Stewartson (1952), Mallick (1957), Sarma (1958) are only particular cases of the results of this paper. Even the particular results of Sinha and Gupta (1979) are erroneous. Results for oblate spheroid are given elsewhere.

1. INTRODUCTION

Many interesting features are presented by the motion of axisymmetric solids in inviscid rotating infinite fluid. It was Taylor (1922) who first undertook the study of a steady motion of a sphere moving along the axis of rotation of an infinite inviscid incompressible fluid. The kinematic conditions at the surface of the sphere and the conditions that the perturbation velocity dies out at infinity could determine only one of the two constants which appeared in his solution, thus leaving the problem indeterminate. It may however be mentioned here that it is possible to remove this type of indeterminacy by introducing viscosity terms in the equations of motion and then letting it to be zero as shown by Rayleigh and later verified by Long. Investigations carried out by Long (1953) on the rotating liquid contained in a circular cylinder have shown that if the ratio of the periphereal velocity of the liquid to the velocity of a sphere ($2\,\Omega a/U$=Rossby number) exceeds a definite limit, waves were produced behind the spherical body and the liquid motions remained indeterminate if further conditions were not imposed. This theoretical deduction seems to be in excellent agreement with experimental observations.

Linearising the equations Stewartson considered the slow uniform motion of a sphere after an impulsive start along the axis of a rotating fluid. The ultimate motion on the surface of the sphere was found to be unsteady; and further a singular surface C, the enveloping cylinder having its generators parallel to the axis of rotation was found to arise which separated the flow in two regions having markedly different characteristics.

Mallick modified the above problem by considering the sphere to be moving with a periodic velocity (set vibrating at $t = 0$) along the axis of rotation using the same linearised equations. A situation similar to that of Stewartson arises; the singular surface C (the enveloping cylinder) is replaced here by C', consisting of two double circumscribing tangent cones, having their vertices on the axis of rotation; further the tangential velocity on the cones attains infinite values.

In part A of this paper we have considered the motion of a prolate spheroid and in part B that of a paraboloid of revolution when each of them is set to vibrate harmonically at $t = 0$. As usual we have based our investigations on the linearised equations of motion, the perturbation being supposed to be small.

We have obtained the solutions in the forms of integrals which permit us to find both a power series expansion or asymptotic expression for the velocity and pressure in two limiting cases. The two cases when $\lambda > 1$ and $\lambda < 1$ where $\lambda =$ (twice the ratio of the angular velocity of the rotating liquid to the frequency of harmonic vibrations of body i.e. $2\,\Omega/\beta$) have been considered. We have focussed our attention on the asymptotic expression for the velocity distribution for large times. In this case the singular surface C (consisting of two similar double tangent cones) which corresponds to the surface C' but with its apex translated. The tangential component of the relative velocity attains infinite values on these cones. The flux normal to the cones is found to be $O(1/\lambda)$; and hence is negligibly small for large values of λ. Then the whole space is divided into eight different zones by the cones three of which are within the volume enclosed by each of these two enveloping cones. Five remain outside any of them. The separation of eight zones with no communicating flow in the limiting case of large λ and infinite velocity components on the cone seems to suggest that under the circumstances envisaged a solution of our problem continuous throughout the entire region of the liquid is not possible on the basis of the linearised equations.

In part B we study the case of a paraboloid of revolution and it is found that no such singular surface as in part A exists here.

The problems which Gupta (1963) has investigated and the results which follow are of a more general nature, and the results of Stewartson, Mallick and Sarma are just particular cases. If in the part A the eccentricity e_1 is made to tend to zero we get the results obtained by Mallick for a sphere and when both e_1 and β approach zero Stewartson's results follow. Letting β tend to zero in part B furnishes the results of Sarma.

2. Formulation of the Problem and its Solutions

Let us suppose that the liquid is rotating about the axis of z with uniform angular velocity Ω, and that the prolate spheroid oscillates along the axis of rotation of the fluid and its velocity at time t is $V(t) = U \cos \beta t$. It is further assumed that the body starts to move under impulse with a velocity $V(0) = U$. The perturbation in the velocities due to the motion of the spheroid should then be sufficiently small for their squares to be neglected.

We take the origin of coordinates at the centre of the spheroid, and adopt cylindrical polar coordinates. Let us take OZ along the axis of rotation and let the components of fluid velocity referred to instantaneously fixed axes along the directions of r, θ, and z, be u, $r\,\Omega+v$ and $w+V(t)$ respectively. We shall take u, v, w, and V to be small quantities where products and higher powers may be neglected. Then if p is the pressure in the fluid, ρ the density and

$$P = \frac{p}{\rho} - \frac{1}{2}\Omega^2 r^2 \qquad \ldots(2.1)$$

the equations of motion then become

$$\frac{\partial u}{\partial t} - 2v\,\Omega = -\frac{\partial P}{\partial r} \qquad \ldots(2.2)$$

$$\frac{\partial v}{\partial t} + 2u\,\Omega = 0 \qquad \ldots(2.3)$$

$$\frac{\partial w}{\partial t} + \frac{\partial V(t)}{\partial t} + \frac{\partial P}{\partial z} = 0. \qquad \ldots(2.4)$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \qquad \ldots(2.5)$$

since the motion is symmetrical about the axis of rotation, and hence independent of θ.

The boundary conditions are that

$$\left.\begin{aligned} &u \to 0;\ v \to 0;\ w = -V(t) \text{ as } z \to \infty \text{ for fixed } r,\ t;\\ &w + \frac{rua^2}{b^2 z} = 0 \text{ on the spheroid } \frac{z^2}{a^2} + \frac{r^2}{b^2} = 1 \text{ for all } t;\\ &w = -V(0) = -U,\ u = 0 = v \text{ when } t = 0 \text{ for all } (r,z) \text{ satisfying } \frac{z^2}{a^2} + \frac{r^2}{b^2} > 1.\end{aligned}\right\} \qquad \ldots(2.6)$$

We shall make use of the Laplace transformation to solve this problem, and introduce the transform such that

$$\bar{w} = \int_0^\infty e^{-st}\, w(r, z, t)\, dt, \text{ etc.} \qquad \ldots(2.7)$$

the function in the transformed space being denoted by a bar.

The equations of motion and continuity become

$$s\bar{u} - 2\Omega\bar{v} = -\frac{\partial \bar{P}}{\partial r} \qquad \ldots(2.8)$$

$$s\bar{v} + 2\Omega\bar{u} = 0 \qquad \ldots(2.9)$$

$$s\bar{w} + s^2\, U/(s^2+\beta^2) = -\frac{\partial \bar{P}}{\partial z} \qquad ...(2.10)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\bar{u}) + \frac{\partial \bar{w}}{\partial z} = 0 \qquad ...(2.11)$$

The boundary conditions would be

$$\bar{u} \to 0 \;\; \bar{v} \to 0; \;\; \bar{w} = -\frac{U_s}{s^2+\beta^2} \;\; \text{as } z \to \infty \text{ for fixed } (r) \qquad ...(2.12)$$

and $\bar{w} + \frac{a^2\,\bar{u}r}{b^2\,z} = 0$ on the spheroid $\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1.$

Solving (2.8) to (2.10) we get

$$\bar{u} = -\frac{s}{s^2+4\,\Omega^2}\frac{\partial \bar{P}}{\partial r}; \;\; \bar{v} = \frac{2\,\Omega}{s^2+4\,\Omega^2}\frac{\partial \bar{P}}{\partial r}; \;\; \bar{w} = -\frac{1}{s}\left(\frac{\partial \bar{P}}{\partial z} + \frac{Us^2}{s^2+\beta^2}\right). \qquad ...(2.13)$$

Substituting these values of $\bar{u}$, $\bar{v}$, $\bar{w}$, in (2.11) we have an equation for $\bar{P}$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{P}}{\partial r}\right) + \frac{s^2+4\,\Omega^2}{s^2}\frac{\partial^2 \bar{P}}{\partial z^2} = 0 \qquad ...(2.14)$$

with the boundary conditions derived from the two equations in (2.12) as follows:-

$$\frac{\partial \bar{P}}{\partial z}; \;\frac{\partial \bar{P}}{\partial r} \to 0 \text{ as } z \to \infty \text{ for fixed } r$$

and

$$\frac{s^2}{s^2+4\,\Omega^2}\cdot\frac{a^2\,r}{b^2\,z}\cdot\frac{\partial \bar{P}}{\partial r} + \frac{\partial \bar{P}}{\partial z} = -\frac{Us^2}{s^2+\beta^2}, \text{ on } \frac{z^2}{a^2} + \frac{r^2}{b^2} = 1. \qquad ...(2.15)$$

Let us now introduce a new set of coordinates defined by

$$z = ck\,\xi\,\eta, \; r = c\,\sqrt{(\xi^2-1)}\,\sqrt{(1-\eta^2)} \qquad ...(2.16)$$

where

$$c^2 = \frac{a^2}{k^2} - b^2\,;\; k^2 = (s^2 + 4\,\Omega^2)/s^2;$$

on the body $\xi = \xi_0 = \dfrac{s}{\sqrt{\{e_1^2\,s^2 - 4\,\Omega^2\,(1-e_1^2)\}}}$,

e_1 being the eccentricity of the elliptic section. Eliminating η from these two equations we have

$$c^2\,k^2\,\xi^4 - \xi^2\,(c^2\,k^2 + r^2k^2 + z^2) + z^2 = 0, \qquad ...(2.17)$$

where

$$2\xi^2 = \frac{(c^2+r^2)\,k^2+z^2 + \sqrt{[\{(r^2+c^2)\,k^2+z^2\}^2 - 4\,c^2k^2z^2]}}{c^2\,k^2} \qquad ...(2.18)$$

Similarly $2\eta^2 = \dfrac{(c^2+r^2)\,k^2+z^2-\sqrt{[\{(r^2+c^2)k^2+z^2\}^2-4c^2\,k^2\,z^2]}}{c^2\,k^2}$ ...(2.19)

The other solutions of (2.18) are not real, when s is real. We shall make ξ one-valued by requiring it to be positive when s is large and positive.
when

$$\frac{r^2}{b^2} \quad \frac{z^2}{a^2} = 1,\ \xi = \xi_0 = \frac{s}{\sqrt{\{e_1^2\,s^2 - 4\,\Omega^2\,(1-e_1^2)\}}}$$

and $\xi \to \infty$ as $z \to \infty$ for fixed r Now

$$\frac{\partial \bar{P}}{\partial \xi} = \frac{z}{\xi}\,\frac{\partial \bar{P}}{\partial z} + \frac{r\,\xi}{\xi^2-1}\,\frac{\partial \bar{P}}{\partial r} \quad \text{...(2.20)}$$

and hence the boundary condition is $\dfrac{\partial \bar{P}}{\partial \xi} = -\dfrac{Us^2ck\eta}{s^2+\beta^2}$ ...(2.21)

Equation (2.14) in terms of (ξ, η) becomes

$$\frac{\partial}{\partial \xi}\left\{(\xi^2-1)\,\frac{\partial \bar{P}}{\partial \xi}\right\} + \frac{\partial}{\partial \eta}\left\{(1-\eta^2)\,\frac{\partial \bar{P}}{\partial \eta}\right\} = 0. \quad \text{...(2.22)}$$

The appropriate solution of this equation is

$$\bar{P} = A\xi\eta + B\eta\xi\left\{\log\frac{\xi+1}{\xi-1} - (2/\xi)\right\}, \quad \text{...(2.23)}$$

where the constants A and B are to be determined by the boundary condition. Since $\dfrac{\partial \bar{P}}{\partial \xi} \to 0$ as $\xi \to \infty$; $A = 0$ since

$$\frac{\partial \bar{P}}{\partial \xi} = -\frac{Us^2ck\eta}{s^2+\beta^2} \text{ at } \xi = \xi_0 = \frac{s}{\sqrt{\{e_1^2\,s^2 - 4\,\Omega^2\,(1-e_1^2)\}}}.$$

We must have

$$B = -\frac{Us^2ck}{s^2+\beta^2}\cdot\frac{1}{\log\dfrac{s+i\,k_1}{s-i\,k_1} - \dfrac{2sik_1}{(1-e_1^2)(s^2+4\,\Omega^2)}} \quad \text{...(2.24)}$$

where $ik_1 = \sqrt{\{e_1^2\,s^2 - 4\,\Omega^2\,(1-e_1^2)\}}$.

$$\text{Hence } \bar{P} = \frac{-U\,s^2\,z\left\{\log\dfrac{\xi+1}{\xi-1} - (2/\xi)\right\}}{(s^2+\beta^2)\left[\log\dfrac{s+i\,k_1}{s-i\,k_1} - \dfrac{2sik_1}{(s^2+4\,\Omega^2)\,(1-e_1^2)}\right]} \quad \text{...(2.25)}$$

where the function which is unique is now completely determined by inverting the transformations with respect to 't' and we deduce that

$$P = -\frac{Uz}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s^2e^{st}}{s^2+\beta^2}\cdot\frac{\left\{\log\dfrac{\xi+1}{\xi-1} - 2/\xi\right\}}{\left[\log\dfrac{s+i\,k_1}{s-i\,k_1} - \dfrac{2sik_1}{(s^2+4\,\Omega^2)\,(1-e_1^2)}\right]}\,ds. \quad \text{...(2.26)}$$

and γ is sufficiently a large and positive quantity. From equation (2.13) we now get formal expressions for u, v, w given by

$$u = -\frac{U}{2\pi i}\int_{\Gamma} \frac{2rzs^{3}e^{st}\left\{\log\frac{(s+ik_1)}{(s-ik_1)} - \frac{2\,sik_1}{(s^2+4\,\Omega^2)\,(1-e_1^2)}\right\}^{-1} ds}{(s) + \beta^2)\,\xi\,(\xi^2-1)\sqrt{[f(r,z,s) - 4a^2z^2s^2\,(k_1 i)]}} \qquad \ldots(2.27)$$

where $f\,(r, z, s) = \left[(r^2+z^2+a^2e_1^2)\,s^2+4\,\Omega^2\,(r^2-a^2+a^2e_1^2)\right]^2$

$$v = \frac{U}{2\pi i}\int_{\Gamma} \frac{s^2\,e^{st}\dfrac{4\,rz\Omega}{\log\dfrac{s+i\,k_1}{s-i\,k_1} - \dfrac{2s\,i\,k_1}{(s^2+4\,\Omega^2)\,(1-e_1^2)}}ds}{(s^2+\beta^2)\,\xi\,(\xi^2-1)\sqrt{\{f\,(r,z,s) - 4\,a^2\,z^2\,s^2\,(k_1\,i)^2\}}} \qquad \ldots(2.28)$$

$$w = -\,U\cos\beta\,t + \frac{U}{2\pi i}\int_{\Gamma} \frac{se^{st}\left[\log\dfrac{\xi+1}{\xi-1} - \dfrac{2\,a^2\xi(k_1\,i)^2}{\sqrt{[f(r,z,s) - 4\,a^2z^2s^2\,(k_1\,i)^2]}}\right.}{(s^2+\beta^2)\left[\log\dfrac{s+k_1\,i}{s-k_1\,i} - \dfrac{2\,s\,k_1\,i}{(1-e_1^2)\,(s^2+)\,4\Omega^2}\right]}\,.$$

$$\ldots(2.29)$$

Equations (2.26), (2.27), (2.28), (2.29) constitute the solutions of the equations (2.8) to (2.11) subject to the given boundary conditions.

3. Pressure and Velocity distribution on the spheroid

The resultant pressure on the spheroid is along the axis of rotation, and is given by

$$Z = -\int\int lpds \qquad \ldots(3.1)$$

where l is the direction cosine of the outward drawn normal in the direction of the Z axis, integral being taken over the surface of the spheroid. Since ξ is constant and is equal to $s/i\,k_1$ on the prolate spheroid, on substituting from (2.1) for p, we have

$$Z = \frac{U}{2\pi i}\int_{\Gamma} \frac{s^2\,e^{st}\left[\log\dfrac{s+i\,k_1}{s-i\,k_1} - \dfrac{2\,ik_1}{s}\right]ds\int\int l\rho z\,ds}{(s^2+\beta^2)\left[\log\dfrac{s+i\,k_1}{s-i\,k_1} - \dfrac{2\,s\,(i\,k_1)}{(s^2+4\,\Omega^2)\,(1-e_1^2)}\right.} \qquad \ldots(3.2)$$

and the integration over the term in (2.1) vanishes. The double integration in (3 2) is equal to $\frac{4}{3}\pi\,\rho a^3\,(1-e_1^2)$ and when Ωt is small, we may obtain a power series for Z by first expanding the logarithmic terms in a series of descending powers of s we get

$$Z = \frac{4}{3}\pi\rho\,a^3\,(1-e_1^2)\,U\left[-\left(\frac{1}{2} - \frac{3\,e_1^2}{10} - \ldots\right)\delta\,(t) + \beta\,\sin\beta t\right.$$

(*equation continued on p.* 1259)

$$\left\{\left(\frac{1}{2}-\frac{3e_1^2}{10}+\dots\right)+\lambda_2^2\left(\frac{3}{10}+\frac{24e_1^2}{125}-\dots\right)-\lambda_2^4\left(\frac{12}{175}+\frac{12e_1^2}{125}-\dots\right)\right.$$
$$\left.-\lambda_2^6\left(\frac{4}{125}-\dots\right)\right\}-\beta^2\lambda_2^2 t\left\{\lambda_2^3\left(\frac{12}{175}+\frac{12}{125}e_1^2-\dots\right)+\lambda_2^5\left(\frac{4}{125}-\dots\right)\right.$$
$$\left.-\beta^4\lambda_2^3\frac{t^3}{4}\left(\frac{4}{125}-\dots\right)-\dots\right] \qquad \dots(3.3)$$

where $\lambda_2 = 2(1-e_1^2)\,\Omega/\beta$, $\delta(t)$ is the Dirac's delta function. For the validity of the above expansion, Ωt should be small.

When Ωt is large, the value of Z can be obtained by inserting cuts in the s-plane from $p = \pm 2i\Omega$ and $s = \pm 2\sqrt{\left(\frac{1-e_1^2}{e_1^2}\right)}\,\Omega$ to infinity along lines on one of which the imaginary part of s is constant and the real part decreases.

The path of integration may be replaced by a path round the infinite semi-circle with $R(s) < 0$ and round the two cuts, together with the contribution from the poles at $s = \pm i\beta$.

Adding all these contributions we have finally

$$Z = -\frac{4}{3}\pi\rho a^3(1-e_1^2)U\left[\beta\sin\beta t\,\frac{\log\frac{1-\lambda_1}{1+\lambda_1}+2\lambda_1}{\left(\log\frac{1-\lambda_1}{1+\lambda_1}+\frac{2\lambda_1}{1-\lambda_1^2}\right)}+\frac{\lambda_1^2\sin 2\Omega t}{(1-\lambda_1^2)\Omega t^2}\right.$$
$$\left.\left\{1+O\left(\frac{\log\Omega t\sqrt{(1-e_1^2)}}{\Omega t\sqrt{(1-e_1^2)}}\right)+O\left\{\frac{1}{\Omega t(1-\lambda_1^2+1)}\right\}\right]\text{for } \lambda_1<1 \quad \dots(3.4)$$

$$Z = -\frac{4}{3}\pi\rho U a^3(1-e_1^2)\left\{\beta\frac{2\pi\lambda_1^3}{\lambda_1^2-1}\cdot\frac{\cos\beta t}{\pi^2+\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}\right)^2}\right.$$
$$+\frac{\beta\sin\beta t\left[\pi^2+\left(\log\frac{\lambda_1+1}{\lambda_1-1}\right)^2-\frac{4\lambda_1}{\lambda_1^2-1}+\frac{2\lambda_1(2-\lambda_1^2)}{\lambda_1^2-1}\log\frac{\lambda_1+1}{\lambda_1-1}\right]}{\pi^2+\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda^2-1}\right)^2}$$
$$\left.-\frac{\lambda^2\sin 2t\Omega}{(\lambda_1^2-1)\Omega t^2}\left[1+O\left(\frac{\log\Omega t\sqrt{(1-e_1^2)}}{\Omega t\sqrt{(1-e_1^2)}}\right)+O\left(\frac{1}{\Omega t(\lambda^2-1)}\right)\right]\right\}\text{ for }\lambda_1>1$$
$$\dots(3.5)$$

where $\lambda_1 = \sqrt{\{\lambda^2(1-e_1^2)+e_1^2\}}$

We note that when $\beta \to 0$ i.e. period of vibration is infinitely large, then $\lambda \to \infty$ and $e_1 \to 0$.

The last expression for Z in (3.5) goes over to Stewartson's result namely,

$$Z = -\frac{16}{3}\pi\rho\Omega a^3 U - \frac{4\pi\rho a^3 U}{3\Omega t^2}\sin 2\Omega t\left\{1+O\left(\frac{\log\Omega t}{\Omega t}\right)\right\} \qquad \dots(3.6)$$

We shall now calculate the velocity components u, v, w, on the spheroid.

$$w = -U\cos\beta t + \frac{U}{2\pi i}\int_\Gamma \frac{s\,e^{st}}{(s^2+\beta^2)}\cdot\frac{\log\frac{s-ik_1}{s+ik_1}+\frac{2ia^2sk_1}{k_3}}{\log\frac{s-ik_1}{s+ik_1}+\frac{2isk_1}{(s+4\Omega^2)(1-e_1^2)}}\,ds$$

where $k_3 = 4\,\Omega^2 z^2 (1 - e_1^2) + a^2 s^2 - e_1^2 z^2 s^2$
As before the branch points at $s = \pm 2\,i\,\Omega$ give contributions which tend to zero as $1/t^2$ as $\Omega\,t \to \infty$. Only the poles at $s = \pm 2\Omega \left[\dfrac{1-e_1^2}{a^2-e^2 z^2}\right]^{\frac{1}{2}} z$ and $s = \pm i\,\beta$ give contributions. The contribution from branch points $s = \pm \dfrac{2\,(1-e^2)^{1/2}\,\Omega}{e_1}$ is zero as can be proved in a similar way as for P and similarly for v and u. Adding up these contributions we find that

$$w = -U\cos\beta t\;\frac{\dfrac{2\lambda_1}{1-\lambda_1^2}+\dfrac{2a^2\lambda_1}{\lambda_1^2 z^2-a^2}}{\log\dfrac{1-\lambda_1}{1+\lambda_1}+\dfrac{2\lambda_1}{1-\lambda_1^2}} + \frac{2a^3 z\lambda\,U(1-e_1^2)}{(a^2-e_1^2z^2)(\lambda_1^2z^2-a^2)}$$

$$\left[\left(\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right)\times\cos 2\Omega z\left(\sqrt{\frac{(1-e_1^2)}{(a^2-e_1^2z^2)}}\right)t+\pi\right.$$

$$\left.\sin 2\Omega z t\sqrt{\frac{(1-e_1^2)}{(a^2-e_1^2z^2)}}\right]\times\frac{1}{\pi^2+\left(\log\dfrac{a+z}{a-z}+\dfrac{2az}{a^2-z^2}\right)^2}$$

for $\lambda_1 < 1$ ...(3.7)

and

$$\left(U\cos\beta t\left[\frac{4\lambda_1^4(a^2-z^2)}{(\lambda_1^2z^2-a^2)(\lambda_1^2-1)}+\frac{2\lambda_1^3(a^2-z^2)}{(\lambda_1^2-1)(\lambda_1^2z^2-a^2)}\log\frac{\lambda_1+1}{\lambda_1-1}\right]\right.$$

$$\left.+\pi U\sin\beta t\;\frac{2\lambda_1^3(a^2-z^2)}{(\lambda_1^2-1)(\lambda_1^2z^2-a^2)}\right)\Bigg/\left(\pi^2+\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}\right)^2\right)$$

$$+\left(\frac{2a^2(1-e_1^2)\lambda_1^2 az\,U}{(a^2-e_1^2z^2)\;(\lambda_1^2z^2-a^2)}\left[\left(\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right)\cos 2\,\Omega zt\sqrt{\left(\frac{(1-e_1^2)}{a^2-e_1^2z^2}\right)}\right.\right.$$

$$\left.\left.+\sin 2\,\Omega t\,z\sqrt{\left(\frac{(1-e_1^2)}{(a^2-e_1^2\;z^2)}\right.}\right]\right)\Bigg/\left(\pi^2+\left(\log\frac{a+z}{a-z}+\frac{2\,az}{(a^2-z^2)}\right)^2\right)$$

for $\lambda_1 > 1$ ...(3.8)

where $\lambda_1^2 = \lambda^2(1-e_1^2)+e_1^2$

Since on the surface of the spheroid $w + \dfrac{a^2}{b^2}\cdot\dfrac{ru}{z} = 0$, the variation of u over the spheroid can be obtained from (3.8) as the case may be. On the surface of the spheroid the variation of v is given by

$$v = \frac{2\,U\Omega irz}{\pi i}\int_{\Gamma}\frac{s\,e^{st}}{s^2+\beta^2}\cdot\frac{k_1^3}{k_3\left\{(s^2+4\,\Omega^2)\;(1-e_1^2)\log\dfrac{s-i\,k_1}{s+i\,k_1}+2isk_1\right\}}\,ds.$$

...(3.9)

As $\Omega t \to \infty$, the only contributions arise from the poles at $s = \pm i\beta$ and $s = 2\Omega iz\sqrt{\left(\frac{1-e_1^2}{(a^2-e_1^2z^2)}\right)}$. Adding up these contributions, we have

$$v = 2\lambda\lambda_1^3 zU\sin\beta t \Big/ (1-\lambda_1^2)(\lambda_1^2z^2-a^2)\left[\log\frac{1-\lambda_1}{1+\lambda_1}+\frac{2\lambda_1}{1-\lambda_1^2}\right]$$

$$+ 4a^3\sqrt{\frac{(1-e_1^2)}{(a-e_1^2z^2)}}\,Urz \Big/ (\lambda_1^2z^2-a^2)(a^2-z^2)$$

$$\times\left(\pi\cos 2\Omega zt\sqrt{\frac{1-e_1^2}{(a^2-e_1^2z^2)}} - \sin 2\Omega tz\sqrt{\frac{(1-e_1^2)}{(a^2-e_1^2z^2)}}\right.$$

$$\left.\times\left[\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right]\right)\Big/\left(\pi^2+\left(\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right)^2\right)$$

$$\text{for } \lambda_1 < 1 \qquad \ldots(3.10)$$

and

$$v = \left(-2Urz\lambda\lambda_1^3\left[\pi\cos\beta t - \left\{\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}\right\}\sin\beta t\right]\right)\Big/$$

$$\left((\lambda_1^2-1)(\lambda_1^2z^2-a^2)\left\{\pi^2+\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}\right)^2\right\}\right)$$

$$+\left(4\lambda^2a^2\sqrt{\frac{(1-e_1^2)rzU}{(a^2-e_1z^2)}}\left\{\pi\cos 2\Omega zt\sqrt{\frac{(1-e_1^2)}{(a^2-e_1^2z^2)}}\right.\right.$$

$$\left.\left.-\left(\sin 2\Omega tz\sqrt{\frac{(1-e_1^2)}{(a^2-e_1^2z^2)}}\right)\left(\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right)\right\}\right)\Big/$$

$$\left((\lambda_1^2z^2-a^2)(a^2-z^2)\left[\pi^2+\left(\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right)^2\right]\right)$$

$$\text{for } \lambda_1 > 1 \qquad \ldots(3.11)$$

It is clear that on the surface of the spheroid u, v, w become infinite when $z = \pm a/\lambda_1$ where $\lambda_1 = \sqrt{(\lambda(1-e_1^2)+e_1^2)}$ for the case $\lambda_1 > 1$.

In the following section the velocity distributions at the ultimate stage, namely when $t \to \infty$ have been discussed. The distributions have been calculated only on the axis of rotation and on specific surface of importance. The motion automatically falls into two categories defined by $\lambda_1 < 1$ and $\lambda_1 > 1$.

4. The Ultimate Velocity Distribution

(a) Velocity on $r = 0$ (axis of rotation).

In this case the roots are those of $s^2 (z^2 - a^2 e_1^2) + 4\,\Omega^2 a^2 (1 - e_1^2) = 0$ and ξ becomes $i z s / a \sqrt{\left\{4\,\Omega^2 (1 - e_1^2) - e_1^2 s^2\right\}}$. The integrals for u and v now are both identically zero and so there is only the axial velocity w.

For this also as before the important contributions to w when Ωt is large arise from the poles at $s = \pm i\beta$ and $s = \pm 2\,ai\,\Omega \sqrt{\left\{\frac{1 - e_1^2}{z^2 \quad a^2 e_1^2}\right\}}$ and we find

$$w = \left[\log\left(\frac{z - a\lambda_1}{z + a\lambda_1}\right)\left(\frac{1+\lambda_1}{1-\lambda_1}\right) - \frac{2\lambda_1 (z-a)(z + a\lambda_1^2}{(1 - \lambda_1^2)(z^2 - a^2\lambda_1^2)}\right] \Big/ \left(\log\frac{1-\lambda_1}{1+\lambda_1} + \frac{2\lambda_1}{1 - \lambda_1^2}\right)$$

$$U \cos\beta t + 2\,Uz^2 a^2 (1 - e_1^2)\lambda^2 \Big/ (z^2 - a^2 e_1^2)(z^2 - a^2\lambda_1^2)$$

$$\times \left[\pi \sin 2\,\Omega t a \sqrt{\left\{\frac{(1 - e_1^2)}{(z^2 - a^2 e_1^2)}\right\}} + \left(\log\frac{z+a}{z-a} + \frac{2az}{z^2 - a^2}\right)\right.$$

$$\left.\cos 2\,\Omega t a \sqrt{\frac{(1 - e_1^2)}{(z^2 - a^2 e_1^2)}}\right] \Big/ \pi^2 + \left(\log\frac{z+a}{z-a} + \frac{2az}{z^2 - a^2}\right)^2$$

$$\text{for } \lambda_1 < 1 \qquad \ldots(4.1)$$

$$w = U\left[\log\left(\frac{z + a\lambda_1}{a\lambda_1 - z}\right)\left(\frac{\lambda_1 - 1}{\lambda_1 + 1}\right) + 2\lambda_1\left(\frac{az}{a^2\lambda_1^2 - z^2} - \frac{1}{\lambda_1^2 - 1}\right)\right]$$

$$\left[\pi \sin\beta t + \left(\log\frac{\lambda_1+1}{\lambda_1 - 1} + \frac{2\lambda_1}{\lambda_1^2 - 1}\right)\cos\beta t\right] \Big/ \left[\pi^2 + \left(\log\frac{\lambda_1 + 1}{\lambda_1 - 1} + \frac{2\lambda_1}{\lambda_1^2 - 1}\right)\right]^2$$

$$+ (2\,a^2 (1 - e_1^2)\, z^2\, U \Big/ (z^2 - a^2 e_1^2)(z^2 - a^2\lambda_1^2)$$

(*equation continued on p.* 1263)

$$\times \left[\left(\log \frac{z+a}{z-a} + \frac{2az}{z^2-a^2} \right) \cos 2\,\Omega\, ta \sqrt{\left\{ \left(\frac{1-e_1^2}{z^2 - a^2 e_1^2} \right) \right\}} \right.$$

$$\left. + \pi \sin 2\Omega\, a\, t \sqrt{\left(\frac{1 - e_1^2}{z^2 - a^2 e_1^2} \right)} \right] \Big/ \left[\pi^2 + \left(\log \frac{a+z}{a-z} + \frac{2\,a\,z}{z^2-a^2} \right)^2 \right]$$

for $\lambda > 1$ ...(4.2)

w becomes infinite at $z = \pm\, a\, \lambda_1$, where λ_1 has the same meaning as before.

(b) Velocity on $r = a\,(1 - e_1^2)^{1/2}$ (Enveloping cylinder) (i) calculation of w:

s = 0 now becomes a branch point of the integral for w, whose contribution is given below. In the counter we insert a cut from $s = 0$ to $s = \rightarrow \infty$ along the negative real axis of the s – plane. We first consider the integral, when $\Omega\, t$ is large; and we find that

$$w = -\, U \cos \beta\, t + \frac{1}{2}\, U \left[\left(\log \frac{\lambda_1+1}{\lambda_1-1} + \frac{2\,\lambda_1}{\lambda_1^2-1} \right) \cos \beta\, t + \pi \sin \beta\, t \right]$$

$$\log \frac{\frac{z}{a\,\lambda_1} + 1 + \left(\sqrt{\left(\frac{4\,z}{a\,\lambda_1} \right)} \sin \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right)}{z/\lambda_1\, a + 1 - 2\sqrt{(z/\lambda_1\, a \sin)}\, (\pi/4 + \theta/2))} \Big/ \left[\pi^2 + \left\{ \log \frac{\lambda_1+1}{\lambda_1-1} + \frac{2\,\lambda_1}{\lambda_1^2-1} \right\} \right]^2$$

$$+ U \left[\tan^{-1} \left(2 \sqrt{(z/a\,\lambda_1)} \cos \left(\frac{\pi}{2} + \frac{\theta}{2} \right) \right) \right] \Big/ (z/a\,\lambda_1 - 1)$$

$$\left\{ \pi \cos \beta\, t - \left(\log \frac{\lambda_1+1}{\lambda_1-1} + \frac{2\,\lambda_1}{\lambda_1^2-1} \right) \sin \beta\, t \right\} \Big/ \left[\pi^2 + \left(\log \frac{\lambda_1+1}{\lambda_1-1} + \frac{2\,\lambda_1}{\lambda_1^2-1} \right)^2 \right]$$

$$+ U \frac{1}{\left[\pi^2 + \left\{ \log \frac{\lambda_1+1}{\lambda_1-1} + \frac{2\,\lambda_1}{\lambda_1^2-1} \right\}^2 \right] \cos \theta} \left\{ - \sqrt{\frac{a\,\lambda_1}{z}} \left[\pi \sin \left(\frac{\theta}{2} + \frac{\pi}{4} + \beta\, t \right) \right. \right.$$

$$\left. \left. + \left(\log \frac{\lambda_1+1}{\lambda_1-1} + \frac{2\,\lambda_1}{\lambda_1^2-1} \right) \right] \cos \left(\frac{\pi}{4} + \frac{\beta\, t}{2} + \frac{\theta}{2} \right) \right\} + U \sqrt{\frac{a}{z}}\, \frac{1}{\pi^2} \int_0^\infty e^{-2\,tx}$$

$$\left[\sqrt{x} - \left(\frac{4}{\pi} + \frac{a^2+z^2}{4\,az} \right) x^{3/2} + \left(\frac{16}{\pi^2} - \frac{e_1^2}{4} \right. \right.$$

$$\left. \left. - \frac{3\,(a^2+z^2)^2}{32a\, z^2} + \frac{a^2+z^2}{\pi az} \right)^2 x^{5/2} + \ldots \right] \frac{d\,x}{x^2 + \frac{1}{\lambda_1^2\,(1-e_1^2)}} \qquad \text{...(4.3)}$$

where $\sin \theta = (a^2 + z^2)/2\, a\, z$ ...(4.3a)

When $\lambda \rightarrow \infty$ then the second term in (4.3) reduces to

$$\frac{U}{\pi^2}\,\pi\tan^{-1}(-0) = \frac{U}{\pi^2}\cdot\pi\cdot\pi = U$$

which cancels the first term in (4.3) in the limit $\beta \to 0$. The third and the fourth terms in (4.3) reduce to

$$\lim_{\beta\to 0} -U\sqrt{\frac{a}{z}}\,\frac{\{4\,\Omega^2(1-e_1^2) - e_1^2\,\beta^2\}^{1/4}}{\pi}\,\sin\left(\frac{\pi}{4}+\beta t\right)/\sqrt{\beta}$$

$$+U\sqrt{\frac{a}{z}}\,\frac{1}{\pi^2}\int_0^\infty e^{-2\alpha_1 tx}\left[x^{-3/2} - \left(\frac{4}{\pi}+\frac{a^2+z^2}{4\,az}\right)x^{-1/2}+O(x^{1/2})\right]dx$$

$$=\lim_{\beta\to 0} -\frac{U}{\pi}\sqrt{\frac{2\,\Omega_1\,a}{z}}\left[\frac{\sin(\pi/4+\beta\,t)}{\sqrt{\beta}} - \frac{1}{\pi}\int_0^\infty \frac{e^{-\alpha t}}{\alpha^{3/2}}\,d\alpha\right.$$

$$\left.+\frac{\left(4/\pi+\dfrac{a^2+z^2}{4\,a\,z}\right)}{2\,\Omega_1\pi}\int_0^\infty \alpha^{-1/2}\,e^{-\alpha t}\,d\alpha + O\left(\int_0^\infty \sqrt{\alpha}\,e^{-\alpha t}\,d\alpha\right)\right]$$

$$=-\frac{U}{\pi}\sqrt{\left(\frac{2\,\Omega_1\,a}{z}\right)}\left[\frac{1}{2\,\pi\,i}\int_\Gamma \frac{e^{st}\,d\,s}{s^{3/2}} + \frac{\left(\dfrac{4}{\pi}+\dfrac{a^2+z^2}{4\,a\,z}\right)}{2\,\pi\,\Omega_1}\right.$$

$$\left.+\sqrt{\frac{\pi}{t}}+0(t^{3/2})\right] = -\frac{U}{\pi}\sqrt{\left(\frac{2\,\Omega_1\,a}{z}\right)}\left[2\sqrt{\frac{t}{\pi}}+O\left(\frac{1}{\Omega_1\sqrt{(\pi\,t)}}\right)\right]$$

$$=-\frac{2\,U}{\pi}\sqrt{\left(\frac{2\,\Omega_1\,a\,t}{\pi\,z}\right)}\left[1+O\left(\frac{1}{\Omega_1\,t}\right)\right]$$

where $\Omega_1 = \Omega\sqrt{(1-e_1^2)}$ ...(4.4)

Hence for $\beta \to 0$ and large $\Omega\,t$ and $e_1 \to 0$ (4.4) reduces to

$$-\frac{2\,U}{\pi}\sqrt{\left(\frac{2\,\Omega\,a\,t}{\pi\,z}\right)}\left[1+O\left(\frac{1}{\Omega\,t}\right)\right]$$

which is in agreement with Stewartson's result.

It is clear that from (4·3) that w becomes infinite when $\cos\theta = 0$ i.e. at the points $z = \pm\, a\left\{(\lambda_1+\sqrt{(\lambda_1^2-1)}\right\}$ on the cylinder $r = a\sqrt{(1-e_1^2)}$ for $\lambda_1 < 1$. The contributions from the branch point $s = 0$ tends to zero for large t, which is quite obvious. In this case when t is large, collecting the contribution from the poles $s = \pm\, i\,\beta$ we have

$$w=U\cos\beta t\;\frac{\log\dfrac{k_1'-1}{k_1'+1} + (2a^2\,\lambda_1^2\,k_1'/(a^2+z^2)\sqrt{\left[1-\dfrac{(2\,a\,\lambda_1\,z^2)}{(a^2+z^2)}\right]}}{\log\dfrac{1-\lambda_1}{1+\lambda_1}+\dfrac{2\,\lambda_1}{1-\lambda_1^2}} \quad ...(4.5)$$

where $k_1' = \sqrt{\left(\frac{a^2 + z^2}{2\, a^2\, \lambda_1}\right)} \left[1 + \left\{1 - \left(\frac{2\, a\, z\, \lambda_1}{a^2 + z^2}\right)^2\right\}^{1/2}\right]^{1/2}$ ...(4.6)

(ii) Calculation of u and v; $s = 0$ is a branch point of the integrands for u and v. Its contribution for large t tends to zero when $\lambda < 1$. In this case non-zero contributions come only from the poles at $s = \pm\, i\, \beta$. These give

$$u = \frac{2\, Uaz\, \cos \beta t}{(a^2 + z^2)\, k_1'^2\, (k_1'^2 - 1)\left\{1 - \left(\frac{2\, az\lambda_1}{a^2 + z^2}\right)^2\right\}^{1/2} \left[\log \frac{1-\lambda_1}{1+\lambda_1} + \frac{2\, \lambda_1}{1-\lambda_1^2}\right]}$$

for $\lambda < 1$ ...(4.7)

$$v = \frac{-\, 2a\, U\, \lambda_1\, z \sin \beta t}{(a^2 + z^2)\, k_1'\, (k_1'^2 - 1)\left\{1 - \left(\frac{2az}{a^2 + z^2}\right)^2\right\}^{1/2}\left\{\log \frac{1-\lambda_1}{1-\lambda_1} + \frac{2\, \lambda_1}{1-\lambda_1^2}\right\}} \text{ for } \lambda < 1$$

...(4.8)

where k_1 has the same meaning as (4.6)

For $\lambda > 1$ the branch point $s = 0$, gives contribution to u and v for large $\Omega\, t$. Proceeding exactly in the same way as for w, we have

$$u = \left(\frac{U}{\cos \theta} \sqrt{\frac{a}{\lambda_1\, z}}\right) \frac{B \cos\left(\beta\, t - \frac{\theta}{2} + \frac{\pi}{4}\right) - C \sin\left(\frac{\pi}{4} + \beta\, t - \frac{\theta}{2}\right)}{B^2 + C^2}$$

$$+ \frac{U}{\pi^2} \sqrt{\left(\frac{a}{z}\right)} \int_0^\infty \frac{e^{-2\Omega t x \sqrt{(1-e_1^2)}}}{x^2 + \frac{1}{\lambda^2\, (1-e_1^2)}} \left\{ x^{3/2} + Ax^{5/2} + O\, (x^{7/2})\right\} d\, x \Big/ \left\{x^2 + \frac{1}{\lambda_2^1\, (1-e_1^2)}\right\}$$

...(4.9)

$$v = \frac{U}{\cos \theta} \sqrt{\frac{a\, \lambda_1}{z}}\; \frac{C \sin\, (\beta\, t\, - \, \theta/2 \, - \, \pi/4) \, - \, B\, (\cos\, \beta\, t\, - \, \theta/2\, - \, \pi/4)}{B^2 + C^2}$$

$$+ \frac{U}{\pi} \sqrt{\left(\frac{a}{z}\right)} \int^\infty \frac{e^{-2\, \Omega\, t\, x \sqrt{(1\, -\, e_1^2)}}}{\left(x^2 + \left(\frac{1}{\lambda^2\, (1\, -\, e_1^2)}\right)\right.} \left\{x^{1/2} + Ax^{3/2} + O\, (x^{5/2})\right\} dx$$

...(4.10)

where

$$A = \frac{z}{a} + \frac{a^2 + z^2}{4\, a\, z} - \frac{4}{\pi}; \; B = \pi\left(1 - \frac{z}{a\, \lambda_1} \sin \theta\right)$$

$$+ \frac{z}{a\lambda_1} \cos \theta \left(\log \frac{\lambda_1 + 1}{\lambda_1 - 1} + \frac{2\, \lambda_1}{\lambda_1^2 - 1}\right)$$

$$C = \left(1 - \frac{z}{a\, \lambda_1} \sin \theta\right) \left(\log \frac{\lambda_1 + 1}{\lambda_1 - 1} + \frac{2\, \lambda_1}{1 - \lambda_1^2}\right) - \frac{\pi\, z}{a\, \lambda_1} \cos \theta \qquad ...(4.11)$$

and θ has the same meaning as before.

As $\lambda_1 \to \infty$, $u \to 0$, $v \to -\frac{2U}{\pi}\sqrt{\left(\frac{2\,\Omega_1 \sqrt{(1-e_1^2)}\,a\,t}{\pi}\right)}$ which reduces to Stewartson's result if $e_1 \to 0$. Further u and v like w become infinite on $r = a\sqrt{(1-e_1^2)}$ at the points $z = \pm\, a\,[(\lambda_1 \pm \sqrt{(\lambda_1^2-1)}]$.

For $r > a\sqrt{\left(1-e_1^2\right)}$ and $r < a\sqrt{\left(1-e_1^2\right)}$ the components of velocity are given by (2.27) to (2.28). The residues at the poles at $s = \pm\,\beta\, i$ of each of the integrand for u, v, and w will contain the factors $\left[(r^2+z^2+a^2e_1^2)-(r^2-a^2-a^2e_1^2)\pm 2\,a\,\lambda_1\,z\right]^{-1/2}$ which becomes infinite only when these factors are zero. In the (r, z) plane these factors give the lines on which the velocity components become infinite as follows :

$$r = \pm\,\frac{z - a\,\lambda_1}{\sqrt{(\lambda_1^2-1)}}\;;\; r = \pm\,\frac{z + a\,\lambda_1}{\sqrt{(\lambda_1^2-1)}}\,. \qquad \ldots(4.12)$$

These lines generate two double tangent cones enveloping the spheroid having vertices on $z \to$ axis (the axis of symmetry) situated at $z = \pm\,a\,\lambda_1$ from the centre of the spheroid. The region or liquid flow is divided into eight parts, five of these lie outside the enveloping cones and the remaining three being confined in the space enclosed by two enveloping cones containing the spheroid.

It is important to note that in the limit $\lambda \to \infty$ there is no flow across the boundary of any two regions of these eight separate zones in which the entire space is divided.

The normal velocity across the line $r = \frac{z - a\,\lambda_1}{\sqrt{(\lambda_1^2-1)}}$ is given by,

$$F = \lim_{r \to \frac{z - a\lambda_1}{\sqrt{(\lambda_1^2-1)}}} \left(\frac{w}{\lambda_1} + \sqrt{\left(\frac{\lambda_1^2-1}{\lambda^2}\right)}\,u\right)$$

$$= -\,\frac{U\cos\beta t}{\lambda_1} + \frac{U}{(2\pi\, i\lambda)}\lim_{r \to \frac{z - a\lambda_1}{\sqrt{(\lambda_1^2-1)}}}\int_{\Gamma}\frac{s e^{st}}{s^2+\beta^2}$$

$$\left[\frac{\left(\log\frac{\xi+1}{\xi-1}\right) ds}{\left[\log\frac{s+\sqrt{(e_1^2 s^2-4\,\Omega^2(1-e_1)}-2s\sqrt{(e_1^2 s^2-4\,\Omega^2(1-e^2)}}{s-\sqrt{\{e_1^2 s^2-4\,\Omega^2(1-e_1^2)(1-e_1^2)(s^2+4\,\Omega^2)}}\right]}\right.$$

$$-\frac{\dfrac{\{2rzs^2(\lambda^2-1)^{1/2}+\xi^2(\xi^2-1)\{e_1 s^2-4\,\Omega^2(1-e_1^2)\}\,]\,ds}{\sqrt{[\{s^2(r^2+z^2+a^2e_1^2)+4\,\Omega^2(r^2-a^2+a^2e_1^2)^2\}^2-4\,a^2z^2s^2(e_1^2s^2-4\,\Omega^2(1-e_1^2)]}}}{\log\dfrac{s+\sqrt{(e_1^2 s^2-4\,\Omega^2(1-e_1^2))}}{s-\sqrt{(e_1^2 s^2-4\,\Omega^2(1-e_1^2))}}-\dfrac{2\,s\sqrt{\{(e_1^2s^2-4\,\Omega^2(1-e_1^2)\}\}}}{(1-e_1^2)(s^2+4\,\Omega^2)}}$$

$$\ldots(4.13)$$

When Ωt is large the only contribution to F arises from the poles $s = \pm i\beta$. Adding up all these contributions we have finally on the line $r = \frac{z - a\lambda_1}{\sqrt{(\lambda^2-1)}}$

$$F = -\frac{U\cos\beta t}{\lambda} + \frac{U\left[\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1-1}\right)\cos\beta t+\pi\sin\beta t\right]}{\lambda\,(1-z/a\,\lambda_1)\sqrt{(z/a\,\lambda_1)}\left[\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}\right)^2+\pi^2\right]}$$

$$+\frac{U\,(\pi\sin\beta t)\left\{\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}+\sqrt{\frac{a\lambda_1}{z}}-\log\frac{1+\sqrt{(z/a\,\lambda_1)}}{1-\sqrt{(z/a\,\lambda_1)}}\right\}}{\lambda\left[\pi^2+\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_0^2-1}\right)^2\right]}$$

$$-U\cos\beta t\,\frac{\left[\left\{\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}\right\}\left\{2\sqrt{\left(\frac{a\lambda_1}{z}\right)}-\log\frac{1+\sqrt{(z/a\,\lambda_1)}}{1-\sqrt{(z/a\,\lambda_1)}}\right\}-\pi^2\right]}{\lambda\left[\pi^2+\left(\log\frac{\lambda_1+1}{\lambda_1-1}+\frac{2\lambda_1}{\lambda_1^2-1}\right)^2\right]}. \quad \text{...(4.14)}$$

It follows from the equation (4.14) that the flux F is $0\left(\frac{1}{\lambda_1}\right)$ and becomes negligibly small for large values of λ; when e_1 tends to zero our results reduce to those of Mallick. Then letting $\lambda \to \infty$ the enveloping cones degenerate into enveloping cylinder of the moving sphere; the problem then reduces to that of Stewartson. The vanishing of the normal velocity across the enveloping cones is in agreement with Stewartson's result.

To describe the features of the secondary motion (u, v, w) we refer it to a system of axis fixed at the centre of the vibrating spheroid. At any point on the axis the secondary motion involves two terms, the first of these corresponds to a synchronous vibration in phase with the spheroid (but of different amplitude) the second arise due to the sort of an interaction between the frequency of the spheroid and the rotational frequency of the entire liquid, and is represented by the term $\cos 2\,\Omega a t\,\sqrt{\left(\frac{1-e_1^2}{z^2-a^2e_1^2}\right)}$

$$= \cos\lambda\,\beta\,t\,\sqrt{\left(\frac{1-e_1^2}{z^2-a^2e_1^2}\right)}$$

The components u, v of the secondary motion show exactly similar feature viz. of vibration in two frequencies namely the frequency of the vibrating spheroid and second one of interaction type mentioned above. But now one notices a difference according as the vibrational character frequency of the spheroid is dominant ($\lambda < 1$) or the rotational frequency is dominant ($\lambda > 1$).

In the first case vibration in the frequency of the spheroid has the same phase as that of the spheroid; while in the second case the phase of this vibration is altered.

5. Part B—Paraboloid of Revolution

In this part we consider the case of a paraboloid of revolution. Let the section of the paraboloid of revolution in the (z, r), plane be $z = - a r^2$. The boundary condition on the paraboloid becomes.

$$2 a r u + w = 0 \text{ or } 2 a r u + w = 0$$

which in view of (2.13) becomes

$$2 a r \frac{s^2}{s^2+4\Omega^2} \frac{\partial \bar{P}}{\partial r} + \frac{\partial \bar{P}}{\partial z} = - \frac{U s^2}{s^2+\beta^2} \qquad \text{...(5.1)}$$

Now we formulate the problem by a very elegant transformation of independent variables in which a paraboloid is a coordinate surface. With this transformation the equation (2.14) is to be solved under the boundary condition (5.1). We introduce a new system of coordinates : (ξ, η) such that

$$z + k^2/4a = k(\xi^2 - \eta^2);\ r = 2\xi\eta \qquad \text{...(5.2)}$$

on the paraboloid

$$\xi = \xi_0 = (k/4a)^{1/2}$$

Making use of this transformation the equation (2.14) becomes

$$\frac{\partial^2 \bar{P}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \bar{P}}{\partial \xi} + \frac{\partial^2 \bar{P}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \bar{P}}{\partial \eta} = 0 \qquad \text{...(5.3)}$$

and the boundary condition becomes

$$\frac{\partial \bar{P}}{\partial \xi} = - 2 U k \xi_0 \frac{s^2}{s^2+\beta^2} \text{ on } \xi = \xi_0.$$

Hence the appropriate solution of the equation (5.3) is

$$\bar{P} = (A + B \log \xi)(C + D \log \eta).$$

and using the boundary condition (5.1) we have

$$\bar{P} = - \frac{2 k V \xi_0^2 s^2}{s^2 + \beta^2} \log \xi \qquad \text{...(5.4)}$$

Now

$$\frac{\partial \bar{P}}{\partial z} = \frac{1}{2k(\xi^2+\eta^2)}\left(\xi \frac{\partial \bar{P}}{\partial \xi} - \eta \frac{\partial \bar{P}}{\partial \eta}\right) = - \frac{U \xi_0^2}{(\xi^2+\eta^2)} \cdot \frac{s^2}{(s^2+\beta^2)}$$

Hence $\frac{\partial \bar{P}}{\partial z} \to 0$ as $\xi \to \infty$, which is in agreement with (2.13) and we get the formal expressions for u, v, w after inverting these Laplace transforms i.e.

$$u = \frac{U}{8 a r \pi i} \int_{\Gamma} \frac{s}{s^2+\beta^2} \left[1 - \frac{(1+4az)s^2+4\Omega^2}{q^2}\right] e^{st} ds$$

$$v = - \frac{U}{4 a r \pi i} \int_{\Gamma} \frac{e^{st}}{s^2+\beta^2} \left[1 - \frac{(1+4az)s^2+4\Omega^2}{q^2}\right] ds \qquad \text{...(5.5)}$$

$$w = -U \cos \beta t + \frac{U}{2 \pi i} \int_{\Gamma} \frac{s}{s^2+\beta^2} \cdot \frac{s^2+4 \Omega^2}{q^2} e^{st} ds.$$

where $\gamma > 0$ and

$$q^2 = \left\{(1+4 a z)^2 + 16 a^2 r^2\right\}^{1/2} \left(s^2 + 4 \Omega^2 \xi_1^2\right) \left(s^2 + 4 \Omega^2 \xi_2^2\right)$$

$$\xi_1^2 = \frac{1 + 4 a z + 8 a^2 r^2 - 8 a r (a z + a^2 r^2)}{(1 + 4 a z)^2 + 16 a^2 r^2}$$

$$\xi_2^2 = \frac{1 + 4 a z + 8 a^2 r^2 + 8 a r (a z + a^2 r^2)}{(1 + 4 a z)^2 + 16 a^2 r^2} \qquad \ldots(5.6)$$

These results constitute the complete solution of the problem. To ascertain the general features of the flow, we consider certain special cases of the formula (5.5).

6. General Features of the Flow

On the surface of the paraboloid $\xi = \xi_0$ the equations represented by (5.5) are simplified considerably and it is easy to find their values.

The integrand has the poles at $s = \pm i \beta$ and $s = \pm \frac{2 i \Omega}{\sqrt{(1+4 a^2 r^2)}}$

The contribution from $s = \pm i\beta$ is

$$\frac{U}{4 a r} \left[1 - \frac{4 \Omega^2 - (1 - 4 a^2 r^2)\beta^2}{(1+4 a^2 r^2) \left(\frac{4 \Omega^2}{1+4 a^2 r^2} - \beta^2 \right)} \right] \cos \beta t$$

or

$$\frac{U}{4 a r} \left[1 - \frac{\lambda^2 - (1-4 a^2 r^2)}{\lambda^2 - (1+4 a^2 r^2)}\right] \cos \beta t$$

The contribution from $s = \pm \frac{2 i \Omega}{\sqrt{(1+4 a^2 r^2)}}$ is

$$- \frac{2 U a r \lambda^2}{(1+4 a^2 r^2)(1+4 a^2 r^2 - \lambda^2)} \cos \frac{2 t \Omega}{\sqrt{(1+4 a^2 r^2)}}$$

Adding all these contributions we have finally

$$u = \frac{U}{4 a r} \left[1 - \frac{\lambda^2 - (1 - 4 a^2 r^2)}{\lambda^2 - (1 + 4 a^2 r^2)}\right] \cos \beta t$$

$$- \frac{1}{(1+4 a^2 r^2)} \cdot \frac{2 U a r \lambda^2}{(1+4 a^2 r^2 - \lambda^2)} \cdot \cos \frac{2 t \Omega}{\sqrt{(1+4 a^2 r^2)}} \qquad \ldots(6.1)$$

We note that when $\beta \rightarrow 0$ (i. e. the period of vibration is infinitely large) so that $\lambda \rightarrow \infty$, these results go over to Sarma's (1958) result viz.

$$u = \frac{2 U a r}{(1+4 a^2 r^2)} \cos \frac{2 t \Omega}{\sqrt{(1+4 a^2 r^2)}} \qquad \ldots(6.2)$$

Similarly

$$v = \frac{2\,a\,r\,U\,\lambda}{\lambda^2 - (1+4\,a^2\,r^2)} \sin\beta\,t + \frac{2\,a\,r\,U\,\lambda^2}{\sqrt{(1+4\,a^2\,r^2)}\,[\sqrt{(1+4\,a^2\,r^2)}\,\lambda^2]}$$

$$\sin\frac{2\,t\,\Omega}{\sqrt{(1+4\,a^2\,r^2)}} \qquad \ldots(6.3)$$

$$w = -\,U\cos\beta\,t + \frac{\lambda^2-1}{\lambda^2-(1+4\,a^2\,r^2)}\,U\cos\beta\,t$$

$$+ \frac{4\,a^2\,r^2\,U\,\lambda^2}{(1+4\,a^2\,r^2)\,(1+4\,a^2\,r^2-\lambda^2)} \cos\frac{2\,t\,\Omega}{\sqrt{(1+4\,a^2\,r^2)}} \qquad \ldots(6.4)$$

which are in agreement with Sarma's (1958) results if $\beta \to 0$ i.e.

$$v = -\,\frac{2\,U\,a\,r}{\sqrt{(1+4\,a^2\,r^2)}} \sin\frac{2\,t\,\Omega}{\sqrt{(1+4\,a^2\,r^2)}} \qquad \ldots(6.5)$$

$$w = -\,\frac{4\,U\,a^2\,r^2}{(1+4\,a^2\,r^2)} \cos\frac{2\,t\,\Omega}{\sqrt{(1+4\,a^2\,r^2)}}$$

Velocity on the axis of rotation $r = 0$ is given by

$$u = 0,\; v = 0$$

$$w = -\,U\cos\beta\,t + \frac{U}{2\,\pi\,i}\int_{\Gamma}\frac{s}{s^2+\beta^2}\cdot\frac{s^2+4\,\Omega^2}{\left(\frac{4\,\Omega^2}{1+4\,a\,z}+s^2\right)(1+4\,a\,z)}\,e^{st}\,d\,s$$

$$= -\,U\cos\beta\,t + \frac{(\lambda^2-1)\,U\cos\beta\,t}{\lambda^2-(1+4\,az)} + \frac{U\,\lambda^2\,4\,a\,z}{(1+4\,a\,z)\,(1+4\,a\,z-\lambda^2)}$$

$$\cos\frac{2\,t\,\Omega}{\sqrt{(1+4\,a\,z)}} \qquad \ldots(6.6)$$

which are also in agreement with Sarma's results if $\beta \to 0$.

ULTIMATE VELOCITY DISTRIBUTION

Now, for large values of t, the integrals of (5.5) may be evaluated by incorporating cuts in the plane from $s = \pm\,2\,i\,\xi_1$ and $s = \pm\,2\,i\,\xi_2$ along the lines on which the imaginary part of s is constant and the real part decreases. The path of integration may be replaced by a path round the infinite semi-circle $R\,(s) < 0$, and contours round the four cuts. For example the contribution from the branch points $s = 2\,i\,\Omega\,\xi_1$ to the integral in the first of the equations (5.5) is found to be

$$e^{2i\Omega\xi_1 t}\left[\frac{\xi_1\,\{(1+4\,a\,z)\,\beta^2 - 4\,\Omega^2\}}{\sqrt{(4\,i\,\Omega\,\xi_1)}\,(4\,\Omega^2\,\xi_1^2-\beta^2)\,(\xi_2^2-\xi_1^2)^{1/2}}\right.$$

$$\left. -\,\frac{(1+4\,a\,z)\,\xi_1}{\sqrt{(4\,i\,\Omega\,\xi_1)}\,\sqrt{(\xi_2^2-\xi_1^2)}}\right]\sqrt{\left(\frac{\pi}{t}\right)} \qquad \ldots(6.7)$$

(*equation continued on p.* 1271)

$$u \sim \frac{U}{16\,(a\,r)^{3/2}\,(a^2\,r^2+az)^{1/4}}\left\{\left[\left(\frac{\xi_1\,\{(1+4\,a\,z)\beta^2-4\,\Omega^2\}}{(4\,\Omega^2\,\xi_1^2-\beta^2)\,\sqrt{\Omega\,\xi_1}}-\frac{(1+4\,a\,z)\,\xi_1}{\sqrt{\Omega\,\xi_1}}\right)\right.\right.$$

$$\sin\left(2\,\Omega\,\xi_1\,t-\frac{\pi}{4}\right)+\left(\frac{\xi_2\,\{(1+4\,a\,z)\,\beta^2-4\,\Omega^2\}}{(4\,\Omega^2\,\xi_2^2-\beta^2)\,\sqrt{\Omega\,\xi_2}}-\frac{(1+4\,a\,z)\xi_2}{\sqrt{\Omega\,\xi_2}}\right)$$

$$\left.\left.\sin\left(2\,\Omega\,\xi_2\,t+\frac{\pi}{4}\right)\right]\right\}\frac{1}{\sqrt{(\pi\,t)}} \qquad \ldots(6.8)$$

$$v \sim -\frac{U}{16\,(a\,r)^{3/2}\,(a^2\,r^2+a\,z)^{1/4}}\left[\left\{\frac{\beta^2\,(1+4\,a\,z)-4\,\Omega^2}{(4\,\Omega^2\,\xi_1^2-\beta^2)\,\sqrt{\Omega\,\xi_1}}+\frac{1+4\,az}{\sqrt{\Omega\,\xi_1}}\right\}\right.$$

$$\cos\left(2\,\Omega\,\xi_1\,t-\frac{\pi}{4}\right)+\left(\frac{\beta^2\,(1+4\,a\,z)-4\,\Omega^2}{(4\,\Omega^2\,\xi_2^2-\beta^2)\,\sqrt{\Omega\,\xi_2}}+\frac{1+4\,a\,z}{\sqrt{\Omega\,\xi_2}}\right)$$

$$\left.\cos\left(2\,\Omega\,\xi_2\,t+\frac{\pi}{4}\right)\right]\frac{1}{\sqrt{(\pi\,t)}} \qquad \ldots(6.9)$$

$$w \sim \frac{U}{4\sqrt{(a\,r)}\,(a^2\,r^2+a\,z)^{1/4}}\left[\left\{\frac{\xi_1\,(4\,\Omega^2-\beta^2)}{(4\Omega^2\xi_1^2-\beta^2)\,\sqrt{\Omega\xi_1^2}}-\frac{\xi_1}{\sqrt{\Omega\,\xi_1}}\right\}\right.$$

$$\sin\left(2\,\Omega\,\xi_1\,t-\frac{\pi}{4}\right)+\left\{\frac{\xi_2\,(4\,\Omega^2-\beta^2)}{(4\,\Omega^2\,\xi_2^2-\beta^2)\,\sqrt{\Omega\,\xi_2}}-\frac{\xi_2}{\sqrt{\Omega\,\xi_2}}\right\}$$

$$\left.\sin\left(2\,\Omega\,\xi_2\,t+\frac{\pi}{4}\right)\right]\frac{1}{\sqrt{(\pi\,t)}}$$

When $\beta \to 0$ all the results go over to Sarma's results.

Here in this case also the amplitude of the oscillations decreases to zero, so that the ultimate motion is in general steady and two-dimensional and the axial velocity of the fluid is ultimately the same as that of the paraboloid.

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