Indian J. pure appl. Math., 13(11) : 1273–1277, November 1982

ON THE QUASI-STATIC THERMAL STRESSES IN A CIRCULAR PLATE

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(*Received* 24 *August* 1981)

The present paper deals with the determination of a quasi-static thermal stresses in a thin circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and the fixed circular edge thermally insulated. The results are obtained in a series form in terms of Bessel functions and they are illustrated numerically.

1. INTRODUCTION

Nowacki (1957) has determined steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge. Further Roy Choudhuri (1972) has succeeded in determining the quasi-static thermal stresses in a thin circular plate subjected to transient temperature along the circumference of a circle over the upper face with lower face at zero temperature and the fixed circular edge thermally insulated.

This paper deals with the more realistic problem (see remarks) of determining the quasi-static thermal stresses in a thin circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and the fixed circular edge thermally insulated. The results presented here will be more useful in engineering problems particularly in the determination of the state of strain in circular plates constituting foundations of containers for hot gases or liquids in foundations for furnaces etc.

2. PROBLEM

Consider a thin circular plate defined by $0 \leqslant r \leqslant a$, $-h/2 \leqslant z \leqslant h/2$. Let the plate be subjected to arbitrary initial temperature over the upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Under these more realistic prescribed conditions the quasi-static thermal stresses in a plate are required to be determined.

The differential equation governing the displacement function $\psi(r, z, t)$ is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = (1 + \nu)\, a_t\, T \qquad \ldots(2.1)$$

with

$$\psi = 0 \text{ at } r = a \text{ for all time } t \qquad \text{...(2.2)}$$

where ν and a_t are Poisson ratio and linear coefficient of thermal expansion of the material of the plate respectively and $T(r, z, t)$ is the temperature of the plate at time t satisfying the equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{K}\frac{\partial T}{\partial t} \qquad \text{...(2.3)}$$

with

$$T = f(r) \text{ at } t = 0 \text{ for } z = h/2 \qquad \text{...(2.4)}$$

$$T = 0 \text{ on } z = -h/2 \qquad \text{...(2.5)}$$

and

$$\frac{\partial T}{\partial a} = 0 \text{ at } r = a \qquad \text{...(2.6)}$$

where K is the thermal diffusivity of the material of the plate.

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \cdot \frac{1}{r}\frac{\partial \psi}{\partial r} \qquad \text{...(2.7)}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \psi}{\partial r^2} \qquad \text{...(2.8)}$$

where μ is the Lamé constant, while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the plate in the plane state of stress.

Equations (2.1) to (2.8) constitute the mathematical formulation of the problem.

3. Solution

To obtain the expression for temperature $T(r, z, t)$, assume

$$T(r, z, t) = \left(z + \frac{h}{2}\right) \sum_{n=1}^{\infty} f_n(t)\, J_0(\alpha_n \cdot r) \qquad \text{...(3.1)}$$

where $\alpha_1, \alpha_2, \ldots$ are roots of the equation $J_1(\alpha a) = 0$, $J_n(x)$ is the Bessel function of the first kind and of order n and the function $f_n(t)$ is yet to be determined.

Equations (2.3) and (3.1) give

$$f_n^1(t) = -\alpha_n^2 K f_n(t). \qquad \text{...(3.2)}$$

Integrating (3.2) we obtain

$$f_n(t) = A_n e^{-\alpha_n^2 K t} \qquad \text{...(3.3)}$$

where A_n is a constant. The constant A_n can be found from the nature of the temperature on the upper face.

Now initially

$$f(r) = h \sum_{n=1}^{\infty} A_n J_0(\alpha_n . r) \qquad ...(3.4)$$

Hence by theory of Bessel functions (3.4) gives

$$A_n = \frac{2/h}{a^2 J_0^2(\alpha_n a)} \int_0^a r J_0(\alpha_n . r) . f(r)\, dr . \qquad ...(3.5)$$

Equations (3.1) and (3.3) give the required expression for $T(r, z, t)$ as

$$T(r, z, t) = \left(z + \frac{h}{2} \right) \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)\, e^{-\alpha_n^2 Kt} \qquad ...(3.6)$$

where A_n is given by (3.5).

Now the displacement function $\psi(r, z, t)$ is obtained from eqns. (2.1) and (3.6) as

$$\psi(r, z, t) = -\left(z + \frac{h}{2} \right) (1 + \nu)\, a_t \sum_{n=1}^{\infty} \frac{A_n J_0(\alpha_n . r)\, e^{-\alpha_n^2 Kt}}{\alpha_n^2} . \qquad ...(3.7)$$

And eqns. (2.7), (2.8) and (3.7) give stress functions as

$$\sigma_{rr} = -2\mu (z + h/2)(1 + \nu)\, a_t \sum_{n=1}^{\infty} \frac{A_n J_1(\alpha_n . r)\, e^{-\alpha_n^2 Kt}}{\alpha_n} \qquad ...(3.8)$$

and

$$\sigma_{\theta\theta} = 2\mu (z + h/2)(1 + \nu)\, a_t \sum_{n=1}^{\infty} \frac{A_n\, e^{-\alpha_n^2 Kt}}{\alpha_n} \times \left[\frac{J_1(\alpha_n r)}{r} - \alpha_n J_0(\alpha_n r) \right] . \qquad ...(3.9)$$

4. Numerical Calculations

Let $f(r) = T_0\, \delta(r - b), (a > b)$ ...(4.1)

where T_0 is constant and $\delta(r)$ is the well-known Dirac's delta function of argument r. Equations (3.5) and (4.1) give

$$A_n = \frac{2bT_0}{a^2 h} \frac{J_0(\alpha_n b)}{J_0^2(\alpha_n a)} . \qquad ...(4.2)$$

Also let $a = 1$, $b = 0.5$, $z = h/2$, then the expression (3.8) and (3.9) yields

$$\frac{\sigma_{rr}}{A} = + \sum_{n=1}^{\infty} \frac{J_0(\alpha_n/2)}{J_0^2(\alpha_n)} . \frac{J_1(\alpha_n . r) . e^{-\alpha_n^2 Kt}}{\alpha_n} \qquad ...(4.3)$$

TABLE I

	r = 0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\left.\frac{\sigma_{rr}}{A}\right\|_{t=0}$	0	8285	15720	21610	25000	25870	24120	19850	14110	7092	0
$\left.\frac{\sigma_{rr}}{A}\right\|_{t=0\cdot5}$	0	15.04	28.55	39.22	45.37	46.97	43.79	36.20	25.63	12.87	0
$\left.\frac{\sigma_{rr}}{A}\right\|_{t=1\cdot0}$	0	0.02744	0.05207	0.07153	0.08275	0.08569	0.07987	0.05244	0.4674	0.02348	0

TABLE II

	r = 0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\left.\frac{\sigma_{\theta\theta}}{A}\right\|_{t=0}$	5815	10640	9581	3178	1349	174 1	−2094	−3775	−4482	−4882	−4677
$\left.\frac{\sigma_{\theta\theta}}{A}\right\|_{t=0\cdot5}$	10.55	19.32	17.40	5.761	2.456	0.3018	−3.802	−6.853	−8.137	−8.865	−8.491
$\left.\frac{\sigma_{\theta\theta}}{A}\right\|_{t=1\cdot0}$	0.01925	0.03522	0.03173	0.01051	0.004488	0.0005505	−0.006934	−0.01250	−0.01485	−0.01617	−0.01549

and

$$\frac{\sigma_{\theta\theta}}{A} = -\sum_{n=1}^{\infty} \frac{J_0(\alpha_n/2)}{J_0^2(\alpha_n)} \frac{e^{-\alpha_n^2 Kt}}{\alpha_n^2} \left[\frac{J_1(\alpha_n r)}{r\,\alpha_n} - J_0(\alpha_n r) \right] \qquad \ldots(4.4)$$

where

$$A = \frac{-2\mu\, T_0\, a_t\,(1-\nu)}{10^5} \qquad \ldots(4.5)$$

On using tables 4, 5 and 7 of Mclachlan (1961), eqns. (4.3) and (4.4) with $K = 0.86$ yield Tables I and II.

5. Remarks

Roy Choudhuri (1972) prescribes the transient temperature on the upper face of the plate which requires the governing system to follow the prescribed transient temperature. On the other hand here only the initial temperature distribution on the upper face of the plate is taken to be known and this is always possible. So the problem considered here is more realistic. Comparing the corresponding results by noting the asymptotic behaviour $\sinh \gamma_n (z + h/2) / \sinh (\gamma_n h) \approx \exp[-\gamma_n (h/2-z)]$, it is observed that the expressions for stress functions obtained are more rapidly convergent than the expressions obtained by Roy Choudhuri. Tables I and II show that stress functions decreases rapidly with time. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (3.8) and (3.9) of stress functions.

References

Mclachlan, N. W. (1961). Bessel Functions For Engineers, second edn. Oxford, pp.215, 216.

Nowacki, W. (1957). The state of stress in a thick circular plate due to a temperature field. *Bull. Acad. Polon. Sci., Ser. Sci. Tech.*, 5, 227.

Roy Choudhuri, S. K. (1972). A note on the quasi-static stress in a thin circular plate due to transient temperature applied along the circumference of a circle over the upper face. *Bull. Acad. Polon Sci., Ser. Sci. Tech.*, 20, 21.