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SOME INHOMOGENEOUS COSMOLOGICAL MODELS IN GENERAL RELATIVITY

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Some inhomogeneous cosmological models of Szekeres type have been obtained for the case of a viscous fluid distribution. Various physical features of such models have been discussed.

1. INTRODUCTION

Inhomogeneous cosmological models play an important role in understanding some essential features of the universe such as the formation of galaxies during its early stage of evolution and the process of homogenization. The early attempts at the construction of such models have been by Tolman (1934) and Bondi (1947) who considered spherically symmetric models. Inhomogeneous plane-symmetric models were considered by Taub (1951, 1956) and later by Tomimura (1978). Szekeres (1975) considered a more general type of orthogonal metric and obtained a class of solutions corresponding to pressure-less perfect fluid. Szafron (1977) extended the work to the case of a perfect fluid with a non-vanishing pressure. Recently Collins and Szafron (1979a, b) and Szafron and Collins (1979) have introduced the concept of intrinsic symmetry to have a systematic study of such models. In the present paper we obtain some cosmological models of Szekeres type corresponding to a viscous fluid. Various physical features of these models have been discussed.

The line-element corresponding to the inhomogeneous universe is taken to be of the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2(dy^2 + dz^2) \qquad \ldots(1.1)$$

where A and B are functions of x, y, z and t. The energy-momentum tensor is taken to be that of a viscous fluid given by (Landau and Lifshitz 1963)

$$T_i^j = (\epsilon + p)\, v_i v^j + p g_i^j - \eta\,(v_{i;}^{j} + v^j{}_{;i} + v^j v^l v_{i;l} + v_i v^l v^j_{;l})$$
$$- \left(\zeta - \frac{2}{3}\eta\right) v^l_{;l}\left(g_i^j + v_i v^j\right) \qquad \ldots(1.2)$$

ϵ being the density, p the pressure, η and ζ coefficients of viscosity assumed to be constant and v^i the flow vector of the fluid satisfying

$$v_i v^i = -1\,. \qquad \ldots(1.3)$$

The coordinates are considered to be comoving so that $v^1 = v^2 = v^3 = 0$ and $v^4 = 1$.

The field equations

$$-8\pi T_i^j = R_i^j - \frac{1}{2} R g_i^j \qquad \text{...(1.4)}$$

lead to

$$8\pi\left[p - 2\eta\frac{A_4}{A} - \left(\zeta - \frac{2}{3}\eta\right)v^l_{;l}\right] = \frac{1}{A^2}\frac{B_1^2}{B^2} + \frac{1}{B^2}\left[\frac{B_{22}}{B} - \frac{B_2^2}{B^2} + \frac{B_{33}}{B} - \frac{B_3^2}{B^2}\right] - \left[\frac{2B_{44}}{B} + \frac{B_4^2}{B^2}\right] \qquad \text{...(1.5)}$$

$$8\pi\left[p - 2\eta\frac{B_4}{B} - \left(\zeta - \frac{2}{3}\eta\right)v^l_{;l}\right] = \frac{1}{A^2}\left[\frac{B_{11}}{B} - \frac{A_1B_1}{AB}\right] + \frac{1}{B^2}\left[\frac{A_{33}}{A} + \frac{A_2B_2}{AB} - \frac{A_3B_3}{AB}\right] - \left[\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB}\right] \qquad \text{...(1.6)}$$

$$8\pi\left[p - 2\eta\frac{B_4}{B} - \left(\zeta - \frac{2}{3}\eta\right)v^l_{;l}\right] = \frac{1}{A^2}\left[\frac{B_{11}}{B} - \frac{A_1B_1}{AB}\right] + \frac{1}{B^2}\left[\frac{A_{22}}{A} - \frac{A_2B_2}{AB} + \frac{A_3B_3}{AB}\right] - \left[\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB}\right] \qquad \text{...(1.7)}$$

$$-8\pi\epsilon = \frac{1}{A^2}\left[\frac{2B_{11}}{B} + \frac{B_1^2}{B^2} - \frac{2A_1B_1}{AB}\right] + \frac{1}{B^2}\left[\frac{A_{22}}{A} + \frac{A_{33}}{A} + \frac{B_{22}}{B} - \frac{B_2^2}{B^2} + \frac{B_{33}}{B} - \frac{B_3^2}{B^2}\right] - \left[\frac{B_4^2}{B^2} + \frac{2A_4B_4}{AB}\right] \qquad \text{...(1.8)}$$

$$\frac{B_{12}}{B} - \frac{B_1B_2}{B^2} - \frac{A_2B_1}{AB} = 0 \qquad \text{...(1.9)}$$

$$\frac{B_{13}}{B} - \frac{B_1B_3}{B^2} - \frac{A_3B_1}{AB} = 0 \qquad \text{...(1.10)}$$

$$\frac{A_{23}}{A} - \frac{A_2B_3}{AB} - \frac{A_3B_2}{AB} = 0 \qquad \text{...(1.11)}$$

$$\frac{B_{14}}{B} - \frac{A_4B_1}{AB} = 0 \qquad \text{...(1.12)}$$

$$\frac{A_{24}}{A} - \frac{A_2B_4}{AB} + \frac{B_{24}}{B} - \frac{B_2B_4}{B^2} = 0 \qquad \text{...(1.13)}$$

$$\frac{A_{34}}{A} - \frac{A_3B_4}{AB} + \frac{B_{34}}{B} - \frac{B_3B_4}{B^2} = 0. \qquad \text{...(1.14)}$$

Eliminating p between (1.5). (1.6) and (1.7), we get

$$\frac{1}{A^2}\left[\frac{A_1B_1}{AB} - \frac{B_{11}}{B} + \frac{B_1^2}{B^2}\right] + \frac{1}{B^2}\left[\frac{B_{22}}{B} - \frac{B_2^2}{B^2} + \frac{B_{33}}{B} - \frac{B_3^2}{B^2} - \frac{A_{33}}{A} - \frac{A_2B_2}{AB} + \frac{A_3B_3}{AB}\right] - \left[\frac{B_{44}}{B} + \frac{B_4^2}{B^2} - \frac{A_{44}}{A} - \frac{A_4B_4}{AB}\right] + 16\pi\eta\left[\frac{A_4}{A} - \frac{B_4}{B}\right] = 0 \qquad \text{...(1.15)}$$

and

$$\frac{A_{22}}{A} - \frac{2A_2B_2}{AB} = \frac{A_{33}}{A} - \frac{2A_3B_3}{AB} . \qquad \ldots(1.16)$$

In the above suffixes 1,2,3,4 after A and B denote differentiation with respect to x, y, z and t respectively.

2. Solutions of the Field Equations

We assume that $B_1 \neq 0$.

From (1.9) and (1.10), we have

$$A = \frac{GB_1}{B} \qquad \ldots(2.1)$$

where G is a function of x and t. From (1.12) and (2.1), we have

$$B = G/H \qquad \ldots(2.2)$$

where H is a function of x, y and z. From (2.1), we get

$$A = G_1 - H_1GH^{-1}. \qquad \ldots(2.3)$$

From (1.11) and (1.16), we have

$$\left(\frac{H_{23}}{H}\right)_1 = 0 \qquad \ldots(2.4)$$

and

$$\left(\frac{H_{22} - H_{33}}{H}\right)_1 = 0. \qquad \ldots(2.5)$$

Since the general solution of the above set is difficult, we consider the special cases :

Case 1—We take

$$H = \left\{\psi(x)\,\phi(y, z)\right\}^{-1}.$$

Equation (1.15) then reduces to

$$-\frac{1}{\phi^2}\left[\left(\frac{\phi_2}{\phi}\right)_2 + \left(\frac{\phi_3}{\phi}\right)_3\right] = L \qquad \ldots(2.6)$$

and

$$\left(\frac{G_1}{G}\right)_{44} + \frac{3G_4}{G}\left(\frac{G_1}{G}\right)_4 + \frac{G_1}{G^3} + 16\pi\eta\left(\frac{G_1}{G}\right)_4 = L\frac{(\psi G)_1}{\psi^3 G^3} \qquad \ldots(2.7)$$

where L is a constant. Integrating (2.7) we get

$$\frac{G_{44}}{G} + \frac{1}{2}\frac{G_4^2}{G^2} - \frac{1}{2G^2} + 16\pi\eta\,\frac{G_4}{G} = -\frac{L}{2\psi^2 G} + l(t) \qquad \ldots(2.8)$$

where l is a function of t alone. Equation (2.6) shows that the 2-space V_2 whose metric is

$$d\Sigma^2 = \phi^2(dy^2 + dz^2) \qquad \ldots(2.9)$$

is of constant curvature L. By suitable transformation of coordinates the metric can be reduced to either of the following forms which correspond to positive, negative and zero curvature of V_2 respectively:

$$ds^2 = -dt^2 + e^{2\lambda}\left[\frac{1}{KR^2+1}\left(1 + \frac{\partial\lambda}{\partial R}R\right)^2 dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \quad \text{...(2.10)}$$

$$ds^2 = -dt^2 + e^{2\lambda}\left[\frac{1}{KR^2-1}\left(1 + \frac{\partial\lambda}{\partial R}R\right)^2 dR^2 + R^2(d\theta^2 + \sin h^2\theta d\phi^2)\right] \quad \text{...(2.11)}$$

$$ds^2 = -dt^2 + e^{2\lambda}\left[\frac{1}{KR^2}\left(1 + \frac{\partial\lambda}{\partial R}R\right)^2 dR^2 + R^2(dy^2 + dz^2)\right] \quad \text{...(2.12)}$$

where λ is a function of R and t satisfying the equation

$$\lambda_{44} + \frac{3}{2}\lambda_4^2 - \frac{Ke^{-2\lambda}}{2} + 16\pi\eta\,\lambda_4 = l(t) \quad \text{...(2.13)}$$

K being a non-zero constant. Metric (2.10) is obviously spherically symmetric and the metric (2.12) is of plane symmetry.

Case 2—Here we assume H to be either of the following forms:

(i) $H = \sin\left\{f(x) + My\right\}$

(ii) $H = \sinh\left\{f(x) + My\right\}$

(iii) $H = \cosh\left\{f(x) + My\right\}$

(iv) $H = a(y^2 + z^2) + by + cz + d$

(v) $H = ae^{My} + be^{-My} + c\sin(Mz + d)$

where a, b, c, d are functions of x and M is a constant. For the choices (i), (ii) and (iii), eqn. (1.15) on integration leads to

$$\frac{G_{44}}{G} + \frac{1}{2}\frac{G_4^2}{G^2} - \frac{M^2+1}{2G^2} + 16\pi\eta\frac{G_4}{G} = l(t). \quad \text{...(2.14)}$$

By suitable transformation of coordinates, the metric (1.1) can be reduced to the following forms corresponding to (i), (ii) and (iii) respectively:

$$ds^2 = -dt^2 + \left\{G_1 - G\cot(X + MY)\right\}^2 dX^2 + G^2\operatorname{cosec}^2(X+MY)(dY^2+dz^2) \quad \text{...(2.15)}$$

$$ds^2 = -dt^2 + \left\{G_1 - G\coth(X+MY)\right\}^2 dX^2 + G^2\operatorname{cosech}^2(X+MY)(dY^2+dz^2) \quad \text{...(2.16)}$$

and

$$ds^2 = -dt^2 + \left\{G_1 - G \tanh(X+MY)\right\}^2 dX^2 + G^2 \operatorname{sech}^2 (X+MY)(dY^2+dz^2). \quad \ldots(2.17)$$

For the choice (iv), eqn. (1.15) on integration leads to

$$\frac{1}{2G^2}(4ad - b^2 - c^2 - 1) + \frac{G_{44}}{G} + \frac{1}{2}\frac{G_4^2}{G^2} + 16\pi\eta\frac{G_4}{G} = l(t). \quad \ldots(2.18)$$

We assume that $4ad = b^2 + c^2 + 1$ and $l(t) = \frac{n}{t^2} + \frac{2}{3}(8\pi\eta)^2$, where n is a constant. Equation (2.18), then gives on integration

$$G = k(x)\, e - 16\pi\eta t/3\; t^{1/3} \cos^{2/3} \log (mt^{-3/2})^A, \text{ for } \frac{2}{3}\eta + \frac{1}{9} < 0$$

$$\left\{\alpha = \sqrt{-(2/3n+1/9)}\right\} \quad \ldots(2.19)$$

$$G = k(x)\, e - 16\pi\eta t/3\; t^{1/3} \cosh^{2/3} \left\{\log (mt^{3/2})\beta\right\} \text{ for } \frac{2}{3}n + \frac{1}{9} > 0$$

where $\left\{\beta = \sqrt{(2/3n+1/9)}\right\}$ ...(2.20)

$$G = k(x)\, e - 16\pi\eta t/3\; t^{1/3} \left\{\log (mt^{3/2})\right\}^{2/3}, \text{ for } \frac{2}{3}n + \frac{1}{9} = 0 \quad \ldots(2.21)$$

where m is a function of x. The metric (1.1) then assumes either of the following forms corresponding to (2.19), (2.20) and (2.21) respectively:

$$ds^2 = -dt^2 + k^2(x)\, e - 32\pi\eta t/3\; t^{2/3} \cos^{4/3}\left\{\log (mt^{-3/2})^\alpha\right\}$$
$$\times \left[\left(\frac{f_1}{f} - \frac{2m_1}{3m}\alpha \tan\left\{\log (mt^{-3/2})^\alpha\right\}\right)^2 dx^2 + \frac{dy^2 + dz^2}{a(y^2+z^2)+by+cz+d}\right] \quad \ldots(2.22)$$

$$ds^2 = -dt^2 + k^2(x)\, e - 32\pi\eta t/3\; t^{2/3} \cosh^{4/3}\left\{\log (mt^{3/2})^\beta\right\}$$
$$\times \left[\left(\frac{f_1}{f} + \frac{2}{3}\frac{m_1}{m}\beta \tanh\left\{\log (mt^{3/2})^\beta\right\}\right)^2 dx^2 + \frac{dy^2 + dz^2}{a(y^2+z^2)+by+cz+d}\right] \quad \ldots(2.23)$$

$$ds^2 = -dt^2 + k^2(x)\, e - 32\pi\eta t/3\; t^{2/3} \left\{\log (mt^{3/2})\right\}^{4/3}$$
$$\times \left[\left\{\frac{f_1}{f} + \frac{2m_1}{3m \log (mt^{3/2})}\right\}^2 dx^2 + \frac{dy^2 + dz^2}{a(y^2 + z^2) + by + cz + d}\right] \quad \ldots(2.24)$$

where $f = \dfrac{k(x)}{a(y^2 + z^2) + by + cz + d}$

For the choice (v) with $m^2 (4ab - c^2) = 1$, eqn. (1.15) gives on integration

$$\frac{G_{44}}{G} + \frac{1}{2}\frac{G_4^2}{G^2} + 16\pi\eta\frac{G_4}{G} = l(t). \qquad ...(2.25)$$

Assuming $l(t) = n/t^2 + 2/3 (8\pi\eta)^2$, we have from (2.25), the values G given by (2.19), (2.20) and (2.21). The metric (1.1) then assumes either of the following forms:

$$ds^2 = -dt^2 + k^2(x)\, e-32\pi\eta t/3\; t^{2/3} \cos^{4/3}\left\{\log(mt^{-3/2})^{\alpha}\right\}$$
$$\times\left[\left(\frac{F_1}{F} - \frac{2}{3}\frac{m_1}{m}\alpha\tan\left\{\log(mt^{-3/2})^{\alpha}\right\}\right)^2 dx^2 + \frac{dy^2+dz^2}{ae^{My}+be^{-My}+c\sin(Mz+d)}\right] \qquad ...(2.26)$$

$$ds^2 = -dt^2 + k^2(x)\, e-32\pi\eta t/3\; t^{2/3} \cosh^{4/3}\left\{\log(mt^{3/2})^{\beta}\right\}$$
$$\times\left[\left(\frac{F_1}{F} + \frac{2}{3}\frac{m_1}{m}\beta\tanh\left\{\log(mt^{3/2})^{\beta}\right\}\right)^2 dx^2 + \frac{dy^2+dz^2}{ae^{My}+be^{-My}+c\sin(Mz+d)}\right] \qquad ...(2.27)$$

$$ds^2 = -dt^2 + k^2(x)\, e-32\pi\eta t/3\; t^{2/3}\left\{\log(mt^{3/2})\right\}^{4/3}$$
$$\times\left[\left\{\frac{F_1}{F} + \frac{2}{3}\frac{m_1}{m\log(mt^{3/2})}\right\}^2 dx^2 + \frac{dy^2+dz^2}{ae^{My}+be^{-My}+c\sin(Mz+d)}\right] \qquad ...(2.28)$$

where $F = \dfrac{k(x)}{ae^{My}+be^{-My}+c\sin(Mz+d)}$.

3. Some Physical Features

Case 1—The pressure and density for the metrics are given by

$$8\pi p = 2l(t) + 8\pi\left(\zeta + \frac{4}{3}\eta\right)\left[3\lambda_4 + \frac{R\lambda_{14}}{1+R\lambda_1}\right]$$

$$8\pi\epsilon = 3\lambda_4^2 + \frac{2R\lambda_4\lambda_{14}}{1+R\lambda_1} - Ke^{-2\lambda}\left[\frac{3+R\lambda_1}{1+R\lambda_1}\right].$$

We see that if $\eta = \zeta = 0$, the pressure is homogeneous. The viscosity therefore introduces inhomogeneity in the pressure term.

If $\eta = \zeta = 0$ and $p = 0$

$$t + t_0(R) = \frac{F(R)}{K^{3/2}}\left[\sqrt{\frac{Ke^{\lambda}}{F}}\sqrt{\frac{Ke^{\lambda}}{F}+1} - \sinh^{-1}\sqrt{\frac{Ke^{\lambda}}{F}}\right] \text{ for } F > 0,\ K > 0.$$

$$t + t_0(R) = -\frac{F(R)}{K^{3/2}}\left[\sqrt{\frac{Ke^{\lambda}}{F}}\sqrt{\frac{-Ke^{\lambda}}{F}-1} + \cosh^{-1}\sqrt{\frac{-Ke^{\lambda}}{F}}\right]$$

for $F < 0$, $K > 0$.

and

$$t + t_0(R) = \frac{F(R)}{(-K)^{3/2}}\left[\sin^{-1}\sqrt{\frac{-Ke^{\lambda}}{F}} - \sqrt{\frac{-Ke^{\lambda}}{F}}\sqrt{\frac{Ke^{\lambda}}{F}+1}\right] \quad \text{for } F > 0,\ K < 0.$$

The density in these cases are given by

$$8\pi\epsilon = \frac{3FR^{-1} + F_1}{e^{3\lambda}\left[\frac{1}{R} + \frac{\sqrt{F}\sqrt{Ke^{\lambda}F^{-1}+1}}{e^{3\lambda/2}}\left\{\frac{\partial t_0}{\partial R} - \frac{F_1}{K^{3/2}}\left(\frac{\sqrt{Ke^{\lambda}F^{-1}}}{\sqrt{Ke^{\lambda}F^{-1}+1}} - \sinh^{-1}\sqrt{Ke^{\lambda}F^{-1}}\right)\right\}\right]} \quad \text{for } F > 0,\ K > 0.$$

$$8\pi\epsilon = \frac{3FR^{-1} + F_1}{e^{3\lambda}\left[\frac{1}{R} + \frac{\sqrt{-F}-\sqrt{-Ke^{\lambda}F^{-1}-1}}{e^{3\lambda/2}}\left\{\frac{\partial t_0}{\partial R} - \frac{F_1}{K^{3/2}}\left(\frac{\sqrt{-Ke^{\lambda}F^{-1}}}{\sqrt{-Ke^{\lambda}F^{-1}-1}} - \cosh^{-1}\sqrt{-Ke^{\lambda}F^{-1}}\right)\right\}\right]} \quad \text{for } F < 0,\ K > 0.$$

and

$$8\pi\epsilon = \frac{3FR^{-1} + F_1}{e^{3\lambda}\left[\frac{1}{R} + \frac{\sqrt{F}\sqrt{Ke^{\lambda}F^{-1}+1}}{e^{3\lambda/2}}\left\{\frac{\partial t_0}{\partial R} - \frac{F_1}{(-K)^{3/2}}\left(\sin^{-1}\sqrt{-Ke^{\lambda}F^{-1}} - \frac{\sqrt{-Ke^{\lambda}F^{-1}}}{\sqrt{Ke^{\lambda}F^{-1}+1}}\right)\right\}\right]} \quad \text{for } F > 0,\ K < 0.$$

(2.10) in this case is a particular case of the inhomogeneous model considered by Bondi (1947). Similarly (2 12) is a special case of plane symmetric inhomogeneous model considered by Tomimura (1978). It can be seen that when λ is a function of t alone, metrics (2.10), (2.11) and (2.12) reduce to Robertson-Walker metric of constant curvature $-K$.

Case 2—The pressure and density for the models (2.15), (2.16) and (2.17) are given respectively by:

$$8\pi p = -2l(t) + 8\pi\left(\zeta + \frac{4}{3}\eta\right)\left[\frac{3G_4}{G} + \frac{\left(\frac{G_1}{G}\right)_4}{\frac{G_1}{G} - \cot(X+MY)}\right]$$

$$8\pi\epsilon = \frac{3G_4^2}{G^2} + \frac{\frac{2G_4}{G}\left(\frac{G_1}{G}\right)_4}{\frac{G_1}{G} - \cot(X+MY)} - \frac{M^2+1}{G^2}\left[\frac{\frac{G_1}{G} - 3\cot(X+MY)}{\frac{G_1}{G} - \cot(X+MY)}\right]$$

$$8\pi p = -2l(t) + 8\pi\left(\zeta + \frac{4}{3}\eta\right)\left[\frac{3G_4}{G} + \frac{\left(\frac{G_1}{G}\right)_4}{\frac{G_1}{G} - \coth(X+MY)}\right]$$

$$8\pi\epsilon = \frac{3G_4^2}{G^2} + \frac{\frac{2G_4}{G}\left(\frac{G_1}{G}\right)_4}{\frac{G_1}{G} - \coth(X+MY)} - \frac{M^2+1}{G^2}\left[\frac{\frac{G_1}{G} - 3\coth(X+MY)}{\frac{G_1}{G} - \coth(X+MY)}\right]$$

and

$$8\pi p = -2l(t) + 8\pi\left(\zeta + \frac{4}{3}\eta\right)\left[\frac{3G_4}{G} + \frac{\left(\frac{G_1}{G}\right)_4}{\frac{G_1}{G} - \tanh(X+MY)}\right]$$

$$8\pi\epsilon = \frac{3G_4^2}{G^2} + \frac{\frac{2G_4}{G}\left(\frac{G_1}{G}\right)_4}{\frac{G_1}{G} - \tanh(X+MY)} - \frac{M^2+1}{G^2}\left[\frac{\frac{G_1}{G} - 3\tanh(X+MY)}{\frac{G_1}{G} - \tanh(X+My)}\right].$$

The pressure p, density ϵ, coefficient of shear σ and the expansion factor θ for the following models are as follows:

For the model (2.22):

$$8\pi p = -\frac{2n}{t^2} + \frac{8\pi\left(\zeta + \frac{4}{3}\eta\right)}{\left[\frac{f_1}{f} - \frac{2}{3}\frac{m_1}{m}\alpha\tan\left\{\log(mt^{-3/2})^\alpha\right\}\right]}$$

$$\times\left[\frac{f_1}{f}\left(-16\pi\eta + \frac{1}{t} + \frac{3\alpha}{t}\tan\left\{\log(mt^{-3/2})^\alpha\right\}\right)\right.$$

$$-\frac{2}{3}\frac{m_1}{m}\alpha\tan\left\{\log(mt^{-3/2})^\alpha\right\}\times\left(-16\pi\eta + \frac{1}{t}\right)$$

$$\left. + \frac{m_1}{mt}\alpha^2\left(1 - \tan^2\left\{\log(mt^{-3/2})^\alpha\right\}\right)\right]$$

$$8\pi\epsilon = \frac{1}{\left[f_1f^{-1} - \frac{2}{3}m_1m^{-1}\alpha\tan\left\{\log(mt^{-3/2})^\alpha\right\}\right]}\times\left[\frac{3f_1}{f}\left(-\frac{16}{3}\pi\eta + \frac{1}{3t}\right.\right.$$

$$\left.+\frac{\alpha}{t}\tan\left\{\log(mt^{-3/2})^\alpha\right\}\right)^2 - \frac{2}{9}\frac{m_1}{m}\alpha\left(-16\pi\eta + \frac{1}{t}\right)^2\tan\left\{\log(mt^{-3/2})^\alpha\right\}$$

$$+\frac{2}{3}\frac{m_1}{m}\alpha^2\left(-16\pi\eta + \frac{1}{t}\right)\left(1 - \tan^2\left\{\log(mt^{-3/2})^\alpha\right\}\right)$$

$$\left.+\frac{2\alpha^3 m_1}{mt^2}\tan\left\{\log(mt^{-3/2})^\alpha\right\}\right]$$

$$\sigma = \frac{1}{\sqrt{3}}\times\frac{m_1(mt)^{-1}\alpha^2\sec^2\left\{\log(mt^{-3/2})^\alpha\right\}}{\left[f_1f^{-1} - \frac{2}{3}m_1m^{-1}\alpha\tan\left\{\log(mt^{-3/2})^\alpha\right\}\right]}$$

$$\theta = \frac{1}{\left[f_1 f^{-1} - \frac{2}{3} m_1 m^{-1} \alpha \tan\left\{\log (m t^{-3/2})^{\alpha}\right\}\right]} \times \left[f_1 f^{-1}\left(-16\pi\eta + \frac{1}{t}\right.\right.$$

$$\left. + 3\alpha t^{-1} \tan\left\{\log (m t^{-3/2})^{\alpha}\right\}\right) - \frac{2}{3}\frac{m_1}{m} \alpha \tan\left\{\log (m t^{-3/2})^{\alpha}\right\}\left(-16\pi\eta + \frac{1}{t}.\right)$$

$$\left. + \frac{m_1}{mt} \alpha^2 \left(1 - \tan^2\left\{\log (m t^{-3/2})^{\alpha}\right\}\right)\right].$$

For the model (2.23):

$$8\pi p = -\frac{2n}{t^2} + \frac{8\pi\left(\zeta + \frac{4}{3}\eta\right)}{\left[f_1 f^{-1} + \frac{2}{3} m_1 m^{-1} \beta \tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right]}$$

$$\times \left[f_1 f^{-1}\left(-16\pi\eta + \frac{1}{t} + \frac{3}{t}\beta \tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right)\right.$$

$$+ \frac{2}{3}\frac{m_1}{m}\beta\left(-16\pi\eta + \frac{1}{t}\right)\tanh\left\{\log (m t^{3/2})^{\beta}\right\}$$

$$\left. + \frac{m_1}{mt}\beta^2\left(1 + \tanh^2\left\{\log (m t^{3/2})^{\beta}\right\}\right)\right].$$

$$8\pi\epsilon = \frac{1}{\left[f_1 f^{-1} + \frac{2}{3} m_1 m^{-1} \beta \tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right]} \times \left[3 f_1 f^{-1}\left(\frac{-16}{3}\pi\eta + \frac{1}{3t}\right.\right.$$

$$\left. + \frac{1}{t}\beta \tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right)^2 + \frac{2}{9}\frac{m_1}{m}\beta\left(-16\pi\eta + \frac{1}{t}\right)^2 \tanh\left\{\log (m t^{3/2})^{\beta}\right\}$$

$$+ \frac{2}{3} m_1 m^{-1} t^{-1}\beta^2\left(1 + \tanh^2\left\{\log (m t^{3/2})^{\beta}\right\}\right)\left(-16\pi\eta + \frac{1}{t}\right)$$

$$\left. + \frac{2\beta^3 m_1}{m t^2}\tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right]$$

$$\sigma = \frac{1}{\sqrt{3}}\frac{m_1 m^{-1} t^{-1}\beta^2 \operatorname{sech}^2\left\{\log (m t^{3/2})^{\beta}\right\}}{\left[f_1 f^{-1} + \frac{2}{3} m_1 m^{-1} \beta \tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right]}$$

$$\theta = \frac{1}{\left[f_1 f^{-1} + \frac{2}{3} m_1 m^{-1} \beta \tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right]} \times \left[f_1 f^{-1}\left(-16\pi\eta + \frac{1}{t}\right.\right.$$

$$\left. + \frac{3}{t}\beta \tanh\left\{\log (m t^{3/2})^{\beta}\right\}\right) + \frac{2}{3} m_1 m^{-1} \beta \tanh\left\{\log (m t^{3/2})\right\}$$

$$\left.\left(-16\pi\eta + \frac{1}{t}\right) + \frac{m_1}{mt}\beta^2\left(1 + \tanh^2\left\{\log (m t^{3/2})^{\beta}\right\}\right)\right]$$

For the model (2.24)

$$8\pi p = -\frac{2n}{t^2} + \frac{8\pi\left(\zeta + \frac{4}{3}\eta\right)}{\left[f_1 f^{-1} + \frac{2}{3}\frac{m_1}{m\log(mt^{3/2})}\right]} \times \left[f_1 f^{-1}\left\{-16\pi\eta + \frac{1}{t} + \frac{3}{t\log(mt^{3/2})}\right\} + \frac{2m_1\left(-16\pi\eta + \frac{1}{t}\right)}{3m\log(mt^{3/2})} + \frac{m_1}{mt\{\log(mt^{3/2})\}^2}\right]$$

$$8\pi\epsilon = \frac{1}{\left[f_1 f^{-1} + \frac{2}{3}\frac{m_1}{m\log(mt^{3/2})}\right]} \times \left[\frac{3f_1}{f}\left\{-\frac{16}{3}\pi\eta + \frac{1}{3t} + \frac{1}{t\log(mt^{3/2})}\right\}^2 + \frac{2}{9}\frac{m_1\left(-16\pi\eta + \frac{1}{t}\right)^2}{m\log(mt^{3/2})} + \frac{2m_1}{3m_1}\left(-16\pi\eta + \frac{1}{t}\right)\Big/\left\{\log(mt^{3/2})\right\}^2\right]$$

$$\sigma = \frac{1}{\sqrt{3}}\frac{\dfrac{m_1}{mt\{\log(mt^{3/2})\}^2}}{\left[\dfrac{f_1}{f} + \dfrac{2m_1}{3m\log(mt^{3/2})}\right]}$$

$$\theta = \frac{1}{\left[f_1 f^{-1} + \frac{2}{3}\frac{m_1}{m\log(mt^{3/2})}\right]} \times \left[\frac{f_1}{f}\left\{-16\pi\eta + \frac{1}{t} + \frac{3}{t\log(mt^{3/2})}\right\} + \frac{2m_1\left(-16\pi\eta + \frac{1}{t}\right)}{3m\log(mt^{3/2})^2} + \frac{m_1}{mt\{\log(mt^{3/2})\}^2}\right].$$

For the models (2.26), (2.27) and (2.28), the expressions for the pressure, density, coefficient of shear and the expansion factor are the same as for (2.22), (2.23) and (2.24) with the only change from f to F respectively.

We thus find that for the models (2.15)-(2.17), (2.22)-(2.24) and (2.26)-(2.28), the pressure is homogeneous in the absence of viscosity. Inhomogeneity in the pressure term is due to viscosity. It is also to be noted that for these models, the viscosity has no effect on the shear. For the models (2.24), (2.25), (2.27) and (2.28) $\sigma \to 0$ as $t \to \infty$. The models (2.23) and (2.27) have singularities at $t = 0$. These models can be considered to be expanding from the singular state provided $\frac{1}{3}\frac{m_1}{m}\beta < \frac{f_1}{f} < \frac{2}{3}\frac{m_1}{m}\beta$ for the model (2.23) and $\frac{1}{3}\frac{m_1}{m}\beta < \frac{F_1}{F} < \frac{2}{3}\frac{m_1}{m}\beta$ for the model (2.28). The expansion for these models ceases after a finite interval of time. However in the absence of viscosity, $\theta \to 0$ as $t \to \infty$ so that the expansion continues upto infinite time. For the models (2.23), (2.24), (2.27) and (2.28)

$$\lim_{t\to\infty}\frac{\sigma}{\theta} = 0.$$

Hence the models approach isotropy for large t. The density contrasts $\frac{\epsilon x}{\epsilon}, \frac{\epsilon y}{\epsilon}, \frac{\epsilon z}{\epsilon}$ tend to non-zero limits for large values of t for the models (2.23) and (2.27) which shows that, in general, inhomogeneity does not die out. However, when a, b, c, d are constants inhomogeneity asymptotically dies out. The density contrast terms for the models (2.24) and (2.28) vanish for large values of t. The models, therefore, approach homogeneity in these cases.

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