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AN EXISTENCE ANALYSIS FOR A FUNCTIONAL DIFFERENTIAL EQUATION WITH PERIODIC BOUNDARY CONDITIONS VIA THE ALTERNATIVE METHOD

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The problem of existence of solutions of boundary value problems for delay differential equations with periodic boundary conditions has been studied by the method of alternative and an indication of the approach to solve concrete problems has been given at the end by putting the delayed arguments in the nonlinear part.

The problems connected with periodic solutions of differential-difference equations and more generally functional differential equations is an important area of mathematical research from practical as well as theoretical points of view. Many authors notably (Hale 1964, Halanay 1961, Jones 1964, Krasnoselskii 1963, Mawhin 1971, Perello 1968, Shimanov 1963) established the existence of periodic and almost periodic solutions for several classes of functional differential equations. The present paper is a contribution in this direction.

In a series of papers initiated by Cesari (1963, 1964), and developed further by Cesari and Hale (1957), Bancroft *et al.* (1968), Knobloch (1963 a,b), Locker (1967, 1970) and others, the method of alternative has come to be recognized as a powerful method for the solution of operator equations of the form $Lx = Nx$, where L is a linear possibly unbounded operator, N a continuous nonlinear operator. This has been applied to the solution of nonlinear BVP, where L is a fairly general differential operator (both self-adjoint and non-self-adjoint). A comprehensive list of such results is found in Cesari (1976).

Locker (1967, 1970) studied in detail, the case of a non-self-adjoint linear ordinary differential operator L on an interval $[a,b]$ with linear boundary conditions at a and b, and N a Nemitsky operator not involving derivatives. These results have been extended to the case where L is a self-adjoint or non-self-adjoint linear ordinary differential operator on an interval $[a,b]$ with multi-point boundary conditions (linear as well as nonlinear) (Venkatesulu 1979).

The basic motivation of the present paper is to extend these results to the case of a nonlinear operator N with delays, where L, as before, is an ordinary differential operator (not necessarily self-adjoint) on $[a,b]$ with periodic boundary conditions.

1. Notations and the Boundary Value Problem

Let α denote a positive real number, J stand for a closed interval $[a,b]$ on the real line with $b > a$ and J_1 stand for the closed interval $[a-\alpha, b]$. δ_{ij} denotes the Kronecker delta of i and j. $D(T)$, $N(T)$, $R(T)$ denote the domain, the null space and the range of the operator T, respectively. $\langle w_1, w_2, \ldots, w_m \rangle$ stands for the linear space spanned by w_1, $w_2, \ldots,$and w_m. $T|_E$ denotes the restriction of the operator T to the set E. R^n stands for the n-dimensional real space with Euclidean norm $|\,.\,|$.

x or $x(.)$ denotes the column vector $x = (x_1, x_2, \ldots, x_n)$ where x_i's are real-valued functions. $x^{(m)}$ denotes the mth derivative of x. C_n^∞ (J) denotes the functions x with coordinates x_i's belonging to $C^\infty(J)$—the space of all infinitely differentiable functions on J. S stands for $L_n^2(J)$—the Hilbert space of all square-integrable functions x on J. The usual inner product and norm in S are denoted by

$$(x, y) = \int_a^b y(t)x(t)dt = \sum_{i=1}^{n} \int_a^b y_i(t)\, x_i(t)\, dt$$

and

$$\| x \| = \sqrt{(x,x)}, \quad \text{where } x, y \in S.$$

I denotes the identity operator on S. $E^\perp$ denotes the orthogonal complement of E in S where E is a subset of S. $E \oplus F$ denotes the direct sum of the subsets E and F of S. $\tilde{S}$ denotes the set of all $x \in S$ with x_i's essentially bounded on J. The norm μ on $\tilde{S}$ is defined by

$$\mu(x) = \max_{i=1,2,\ldots,n} \operatorname*{ess.sup}_{t \in J} | x_i(t) | , \; x=(x_1,\ldots,x_n) \in \tilde{S}.$$

$H(J)$ denotes the linear subspace of S consisting of all functions x whose components x_i's are absolutely continuous on J and $x^{(1)} \in S$. We take $L_n^2[-\alpha,0]$ to be the Hilbert space of all functions $g[=(g_1,\ldots,g_n)$-column vector] which are square-integrable on $[-\alpha, 0]$ with the usual inner product and norm denoted through $(.,.)_1, \|.\|_1$, respectively.

We consider the following system of n first order nonlinear functional differential equations with periodic boundary conditions:

$$\Omega x = x^{(1)} - A(t)x = X(t,x_t) \qquad \ldots(1.1)$$

$$x_a = x_b \qquad \ldots(1.2)$$

where $A(t)$ is an $n \times n$ matrix function composed of C^∞ functions on J and $x_t(\theta) = x(t+\theta)$, $-\alpha \leqslant \theta \leqslant 0$, for every $t \in J$. We assume that our nonlinear function $X(t,x_t)$ $[X = (X_1, X_2,\ldots,X_n)$-column vector] satisfies the following: (i) $X(t,g)$ is defined for $t \in J$ and $g \in L_n^2\,[-\alpha, 0]$ with $\max_{i=1,2,\ldots,n} \operatorname*{ess.sup}_{\theta \in[-\alpha,0]} | g_i(\theta) | \leqslant R$ where $R > 0$ is a real number.

(ii) $X(.,g) \in S$ for each fixed g.
(iii) There exist real numbers $k_i \geqslant 0$ $(i = 1,2, \quad ,n)$ such that

$$| X_i(t,\bar{g}) - X_i(t,\bar{\bar{g}}) | \leqslant k_i\Big(\sum_{j=1}^{h}\sum_{l=1}^{n} | \bar{g}_1(\theta_j) - \bar{\bar{g}}_1(\theta_j)|^2\Big)^{1/2}, \ \theta_1,\theta_2,\ldots,\theta_h \in [-\alpha,0].$$

2. Setting of the Original Problem as an Operator Equation in the Spaces and Some Properties

For the operator Γ and the boundary forms $B(x) \equiv x(a) - x(b)$, the differential operator $L:D(L) \subset S \to R(L) \subset S$ is defined as follows:

$$\left.\begin{aligned} D(L) &= \{x \in H(J) : x(a) = x(L)\} \\ Lx &= \Gamma x. \end{aligned}\right\} \qquad \ldots(2.1)$$

The operator L has the following well known properties:
(i) $D(L)$ is dense in S.
(ii) L is a closed linear operator.
(iii) $R(L)$ is closed in S.
(iv) $S = R(L) \oplus N(L)$, where L denotes the adjoint of L.
(v) $P = \dim N(L) < \infty$, $q = \dim N(L) < \infty$. In fact, $p = q \leqslant n$.
For a method of proof of all these statements we can refer to Brown, MRC Technical Summary Report and the references given on there.

We know that the null space of Γ is n-dimensional with $N(L) \subset N(\Gamma)$. Choose functions $\phi_1, \phi_2,\ldots,\phi_n \in C_n^\infty(J)$ to form an orthonormal base for $N(\Gamma)$ in such a way that $\phi_1, \phi_2,\ldots,\phi_p$ form an orthonormal base for $N(L)$. We note that the restriction of L to the subspace $D(L) \cap N(L)^{\perp}$ is a closed $1-1$ operator and that its inverse

$$H = [L \mid D(L) \cap N(L^{\perp})]^{-1}$$

is a $1-1$ continuous linear operator (by the closed Graph Theorem) with domain $R(L)$, range $D(L) \cap N(L)^{\perp}$, and

$$LHy = y \text{ for all } y \in R(L), \qquad \ldots(2.2)$$

$$HLx = x - \sum_{i=1}^{p} (x,\phi_i)\phi_i \text{ for all } x \in D(L). \qquad \ldots(2.3)$$

Besides, H is a continuous right inverse of L. We next define the sequences of projections $\{P_m\}$ and $\{Q_m\}$ in S and study their relations with L and H.

We choose functions $\omega_1, \omega_2,\ldots,\omega_p$ to form an orthonormal base for $N(L)$. Let $m \geqslant p$ be any integer, and choose functions $\omega_{p+1}, \omega_{p+2}, \ldots,\omega_m$ in $D(L)$ such that the functions $\omega_1, \omega_2, \ldots, \omega_m$ form an orthonormal set in S. Note that the functions

$\omega_{p+1}, \omega_{p+2},\ldots,\omega_m$ belong to $R(L)$, and hence, we can form the functions $H\omega_{p+1},\ldots,H\omega_m$ which belong to $D(L) \cap N(L)^{\perp}$.

Let S_0 be the m-dimensional subspace of S spanned by the functions $\phi_1, \phi_2,\ldots,\phi_p$ and $H_{p+1},\ldots,H_m$. Clearly, $S_0 \subset D(L)$. Let P_m and Q_m be the projection operators defined in S by

$$Q_m x = \sum_{i=1}^{m} (x,\omega_i)\omega_i \text{ for all } x \in S,$$

and

$$P_m x = \sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=p+1}^{m} (x,L\,\omega_i)\, H\omega_i \text{ for all } x \in S.$$

The operators P_m and Q_m have the following properties :

(i) P_m and Q_m are continuous linear operators defined on all of S.

(ii) $R(Q_m) = \left\langle \omega_{1,2}, \ldots, \omega_m \right\rangle$.

(iii) $R(P_m) = S_0$.

(iv) $P_m^2 = P_m'$ and $Q_m^2 = Q_m$

(v) The range of $(I-Q_m)$ is a subset of $R(L)$ and $H(I-Q_m)$ is a continuous linear operator defined on all of S.

The operators P_m, Q_m, L, H satisfy some properties which are summarized in the following theorem :

Theorem 2.1—The following properties are valid :

(i) $H(I - Q_m)Lx = (I - P_m)x$ for all $x \in D(L)$.

(ii) $LH(I - Q_m)x = (I - Q_m)x$ for all $x \in S$.

(iii) $LP_m x = Q_m Lx$ for all $x \in D(L)$.

(iv) $P_m H(I - Q_m)\, x = 0$ for all $x \in D(L)$.

These identities are immediate consequence of the definitions of the operators L, H, P_m, Q_m and are essentially the same as those requested by Cesari (1963, 1964, 1976), and restated by Locker (1967, 1970).

We next show that H and $H(I-Q_m)$ have integral representations.

Let $\phi(t)$ be the $n \times n$ fundamental matrix for $\tau \uparrow$ whose n columns are $\phi_1, \phi_2,\ldots,\phi_n$. Let $G(.,\ldots)$ be the matrix function defined on $J \times J$ by

$$G(t,s) = \phi(t)\, \phi^{-1}(s) \text{ for } a \leqslant s \leqslant t \leqslant b,$$
$$= 0 \qquad \text{for } a \leqslant t \leqslant s \leqslant b. \qquad \ldots(2.4)$$

We need the following lemmas :

Lemma 2.1—Let $y \in S$ and let

$$u(t) = \int_a^t G(t,s)\, y(s)ds, \quad t \in J. \qquad \ldots(2.5)$$

Then the function $u \in H(J)$ and $\Gamma(u) = y$.

For a proof of the lemma see Theorem 3.1 of Chapter 3 of Coddington and Levinson (1972).

Lemma 2.2—There exist real numbers a_{lj}, $l = p+1,\ldots,n$; $j = 1,2,\ldots,n$ such that $\sum_{j=1}^{n} a_{lj}B_j(\phi_i) = \delta_{il}$, for $l,i = p+1,\ldots,n$. Here B_j's are defined by

$$B_j(x) \equiv x_j(a) - x_j(b), \; [x = (x_1,\ldots,x_n)].$$

PROOF: Let D be the $n \times (n-p)$ matrix with entries $B_j(\Phi_i)$, $j = 1, 2, \ldots, n$; $i = p+1,\ldots,n$. It is easily seen that D has rank $n-p$. Consider a suitable $n \times p$ matrix with linearly independent columns such that these columns are linearly independent with that of D and denote the matrix by D_1. Let $(D{:}D_1)$ be the $n \times n$ matrix formed by the elements of D and D_1 such that the elements of D occupy the first position. Clearly, $(D{:}D_1)$ is nonsingular. Hence, $(D{:}D_1)$ has an inverse, say A. This matrix gives the required numbers a_{lj}s.

The following theorem gives an integral representation for H (Venkatesulu 1979).

Theorem 2.2—Let $y \in R(L)$. Then Hy has a representation given by

$$(Hy)(t) = \sum_{l=1}^{n} \Phi_i(t) \int_a^b \psi_l(s)y(s)ds + \int_a^t G(t,s)y(s)ds, \quad t \in J$$

where $\psi_l(s)$ are defined as follows:

$$\psi_l(s) = \begin{cases} -\int_s^b \Phi_l(t)\Phi(t)\Phi^{-1}(s)dt, \; l = 1,2,\ldots,p; & \ldots(2.6) \\ (a_{lj})_{j=1}^{n} (\Phi(b)\Phi^{-1}(s)), \; l = p+1, p\ldots,n. & \ldots(2.7) \end{cases}$$

We note that each ψ_l is a row vector with n components. Let $\psi_l = (\psi_{l1},\psi_{l2},\ldots,\psi_{ln})$. Let Φ_1 be the $n \times n$ matrix with ψ_i occupying the throw. Let $K(.,.)$ denote the $n \times n$ matrix function defined on the square $J \times J$ by

$$K(t,s) = \begin{cases} \Phi(t)\Phi_1(s) + \Phi(t)\Phi^{-1}(s), & a \leqslant s \leqslant t \leqslant b; \\ \Phi(t)\Phi_1(s), & a \leqslant t \leqslant s \leqslant b. \end{cases} \qquad \ldots(2.8)$$

We note that $K(.,s)$ is continuous on J except at the point $t = s$.
Consequently by the above theorem we have

Corollary: The right inverse operator H has an integral representation given by

$$(Hy)(t) = \int_a^b K(t,s)\, y(s)ds, \quad t \in J,$$

for all $y \in R(L)$.

Let $K_m(.,.)$ be the matrix function defined on the square $J \times J$ by

$$K_m(t,s) = K(t,s) - \sum_{i=1}^{m} (\omega_i, K(t,.)^*)\omega_i^*(s), \ a \leqslant t, \ s \leqslant t. \qquad \ldots(2.9)$$

Denote

$$K_m(t,s) = \begin{cases} K_{m,11}(t,s) \ K_{m,12}(t,s) \ldots K_{m,1n}(t,s) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ K_{m,n1}(t,s) \ K_{m,2}(t,s) \ K_{m,nn}(t,s) \end{cases}$$

We notice that $K_{m,ij}(t,s)$, $i,j = 1,2,\ldots,n$ are square integrable on $J \times J$, while the functions $\int_a^b K^2_{m,ij}(t,s)\,ds$, $i,j = 1,2,\ldots,n$ are continuous on J.

The following integral representation for $H(I-Q_m)$ is now an immediate consequence of the above relations.

Theorem 2.3—Let $x \in S$. Then

$$(H(I-Q_m)x)(t) = \int_a^b K_m(t,s)x(s)ds, \ t \in J.$$

Let x be a function on $J = [a, b]$. The function x is extended periodically to a function $\tilde{x}$ on $J_1 = [a-\alpha, b]$ as follows:

$$\tilde{x}(t) = x(t) \qquad \text{for } t \in [a,b]$$
$$= x(t+b-a) \text{ for } t \in [a-, \alpha, a].$$

The operator N is defined as below:

$$D(N) = \{x \in \tilde{S}: \max_{i=1,\ldots,n} \text{ess} \sup_{t \in J} | x_i(t) | \leqslant R\}, \ x = (x_1,\ldots,x_n);$$

$$(Nx)(t) = X(t, \tilde{x}_t), \ t \in J.$$

It is easily seen that

$$Lx = Nx \qquad \ldots(2.10)$$

has a solution x on J if and only if eqns. (1.1)—(1.2) have a solution $\tilde{x}$.

We end up this section by deriving some estimates which are useful later. Let $x, y \in D(N)$. Then, by (1.3)

$$\| Nx-Ny \| = \| X(.,\tilde{x})-X(.,\tilde{y}) \|$$
$$= (\int_a^b \sum_{i=1}^{n} | X_i(t, \tilde{x}_t) - X_i(t, \tilde{y}_t) |^2 dt)^{1/2}$$

(*equation continued on p.* 1302)

$$\leqslant\Big(\int_a^b\sum_{i=1}^{n} k_i^2\sum_{j=1}^{h}\sum_{l=1}^{n} |\tilde{x}_l(t+\theta_j)-\tilde{y}_l(t+\theta_j)|^2dt\Big)^{1/2}$$

$$=\Big(\sum_{i=1}^{n} k_i^2\Big)^{1/2}\Big(\sum_{j=1}^{h}\sum_{l=1}^{n}\int_a^b |\tilde{x}_l(t+\theta_j)-\tilde{y}_l(t+\theta_j)|^2dt\Big)^{1/2}.$$

Also, by the definition of $\tilde{x}$, we have

$$\int_a^b |\tilde{x}_l(t+\theta_j)-\tilde{y}_l(t+\theta_j)|^2dt=\int_a^b |x_l(t)-y_l(t)|^2dt, \qquad j=1,2,\ldots,h.$$

Hence,

$$\| Nx-Ny \| \leqslant \Big(\sum_{i=1}^{n}k_i^2\Big)^{1/2}\Big(h\int_a^b\sum_{l=1}^{n} |x_l(t)-y_l(t)|^2dt\Big)^{1/2}$$

$$= h^{1/2}\Big(\sum_{i=1}^{n}k_i^2\Big)^{1/2}\| x-y \|$$

Take $k_0 = h^{1/2}\Big(\sum_{i=1}^{n}k_i^2\Big)^{1/2}$. Then

$$\| Nx-Ny \| \leqslant k_0 \| x-y \|. \qquad \ldots(2.11)$$

Let $x \in S$. Then

$$\Big\| \int_a^b K_m(\cdot,s)x(s)ds \Big\| =\Big(\sum_{i=1}^{n}\Big(\int_a^b\Big(\int_a^b\sum_{j=1}^{n}K_{m,ij}(t,s)x_j(s)ds\Big)^2dt\Big)^{1/2}$$

$$\leqslant\Big(\sum_{i=1}^{n}\Big(\int_a^b\int_a^b\sum_{j=1}^{n}K_{m,ij}^2(t,s)ds\Big)\| x \|^2dt\Big)^{1/2}$$

$$=\Big(\int_a^b\int_a^b\sum_{i=1}^{n}\sum_{j=1}^{n}K_{m,ij}^2(t,s)ds\,dt\Big)^{1/2}\| x \|.$$

Denote by

$$\theta_m=\Big(\int_a^b\int_a^b\sum_{i=1}^{n}\sum_{j=1}^{n}K_{m,ij}^2(t,s)ds\,dt\Big)^{1/2}.$$

Then

$$\| \int_a^b K_m(.,s)x(s)ds \| \leqslant \theta_m \| x \|. \quad ...(2.12)$$

Similarly one can show that

$$\mu(\int_a^b K_m(.,s)x(s)ds) \leqslant \bar{\theta}_m \| x \| \quad ...(2.13)$$

where

$$\bar{\theta}_m = \max_{i=1,\ldots,n} (\sup_{t \in J} \int_a^b \sum_{j=1}^n K^2_{m,ij}(t,s)ds)^{1/2}.$$

3. Existence Theory

In this section we develop the existence theory to the equation $Lx = Nx$.

Let us choose $x_0 \in S_0$ such that $\beta = \mu(x_0) \leqslant R$. Let $z_0 = H(I - Q_m)Nx_0$ and let e and $\bar{e}$ be real constants such that $\| z_0 \| \leqslant e$ and $\mu(z_0) \leqslant \bar{e}$. ...(3.1)

Let c, d, r and $\bar{R}$ be positive real numbers such that

$$c + e < d, \bar{R} + \beta \leqslant R, \text{ and } r + \bar{e} < R. \quad ...(3.2)$$

The sets V and $\tilde{S}_0$ in S are defined as follows:

$$V = \{x \in S_0 : \| x - x_0 \| \leqslant c,\ \mu(x - x_0) \leqslant r\}, \quad ...(3.3)$$

$$\tilde{S}_0 = \{x \in \tilde{S} : \| x - x_0 \| \leqslant d,\ \mu(x - x_0) \leqslant \bar{R}\}. \quad ...(3.4)$$

Clearly, $x_0 \in V \subset \tilde{S}_0 \subset D(N)$. Moreover, V and $\tilde{S}_0$ are closed, bounded subsets of S.

For each $x^* \in V$, let T be the operator on $\tilde{S}_0$ defined by

$$Tx = x^* + H(I - Q_m)Nx, \quad ...(3.5)$$

for all $x \in \tilde{S}_0$. We observe that T is well defined on $\tilde{S}_0$.

Suppose $x = Tx = x^* + H(I - Q_m)Nx$ fot some $x \in \tilde{S}_0$, ...(3.6)

Clearly, $x \in D(L)$, and by Theorem 2.1 (iv) we have $P_m x = x^*$.

Thus

$$Lx = L\,P_m x + LH(I - Q_m)Nx. \quad ...(3.7)$$

Using parts (ii) and (iii) of Theorem 2.1. we get

$$Lx - Nx = Q_m\,(Lx - Nx). \quad ...(3.8)$$

Hence $x \in \tilde{S}_0$ is a solution of (2.13), if it satisfies the equation

$$Q_m (Lx - Nx) = 0. \qquad ...(3.9)$$

Equation (3.6) is called the auxiliary equation and (3.9) is called the bifurcation equation.

Therefore, for proving the existence of solution of the original problem it is enough to solve equations

$$x = x^* + H (I - Q_m) Nx, \qquad ...(3.10)$$

and

$$Q_m (Lx - Nx) = 0.$$

Define $\psi x = Q_m (Lx - Nx)$.

We solve the above equations by the method used us in Venkatesulu (1979). Below, we present two standard results based on Brouwer's fixed point theorem which establish the existence of solution of system (3.10).

The following considerations which help in the reduction of the problem to Euclidean space are needed before we actually state these theorems.

Apply the Gram-Schmidt process to the functions $H\omega_{p+1},\ldots,H\omega_m$ to obtain orthonormal functions $\eta_{p+1},\ldots,\eta_m$. Assume $M = m$, and let E^M be a copy of Euclidean M-space where we represent each point $\xi \in E^M$ as an M-tuple: $\xi = (b_1,b_2,\ldots,b_p, c_{p+1},\ldots,c_m)$. We define two isomorphisms: $\Gamma_1 : E^M \to S_0$ and $\Gamma_2 : \langle \omega_1,\ldots,\omega_m \rangle \to E^M$ by

$$\Gamma_1(b_1,b_2,\ldots,b_p,\ c_{p+1},\ldots,c_M) = \sum_{i=1}^{p} b_i\Phi_i + \sum_{i=p+1}^{M} c_i\eta_i$$

and

$$\Gamma_2\left(\sum_{i=1}^{M} u_i\omega_i\right) = (u_1,\ u_2,\ldots,u_M).$$

Let $U ; E^M \to E^M$ be the operator defined by $= \Gamma_2\, \psi\, \Gamma_1$.

Choose a function $x_0 \in S_0$. In most applications x_0 is chosen such that $\psi(x_0) = 0$. Let $\xi_0 \in E^M$ be the element with $\Gamma_1(\xi_0) = x_0$.

The following theorems establish the existence of solution of (3.10) and hence of (1.1) - (1.2).

Theorem 3.1 – Let $m = 1$ and conditions (3.1) and (3.2) be satisfied. Let the following conditions be satisfied:

(a) $\theta_m k_0 < 1$, $c + e \leqslant (1 - \theta_m k_0)d$, $r + \bar{e} \leqslant \bar{R} - \bar{\theta}_m k_0 d$;

(b) the set $U = \{\xi \in E^M \mid \xi - \mid \xi_0 \mid \leqslant \epsilon$ is mapped by Γ_1 into the set V with $\epsilon > 0$

(c) the interval $[-\delta, \delta]$ is a subset of $\psi(U)$ and $(\theta_m k_0 d + e)k_0 \leqslant \delta$ with $\delta > 0$; then there exists a function $x \in D(L) \cap D(N)$ which is a solution of the equation $Lx = Nx$. Moreover, $\| x - x_0 \| \leqslant d$ and $\mu(x - x_0) \leqslant \bar{R}$.

Proof of the theorem is a straight forward application of the intermediate value theorem for continuous functions. The next theorem is valid for $m \geqslant 1$.

Theorem 3.2—Let $\psi(\xi_0) = 0$ and the conditions (3.1) and (3.2) be satisfied.

(a) $\theta_m k_0 < 1$, $c+e \leqslant (1-\theta_m k_0)d$, $r+\bar{e} \leqslant \bar{R}-\bar{\theta}_m k_0 d$;

(b) the set $U\{\xi \in E^M : |\xi-\xi_0| \leqslant \epsilon\}$ is mapped by Γ_1 into V with $\epsilon > 0$, the first order partial derivatives of μ exist and are continuous on U, and the Jacobian matrix for ψ has rank m at ξ_0;

(c) the set $W = \{u \in E^M : |u| \leqslant \delta\}$ is a subset of (U), there exists a continuous mapping $\Lambda : W \to U$ with $\psi(\Lambda(u)) = u$ for all $u \in W$, and $(\theta_m k_0 d + e)k_0 < \delta$ with $\delta > 0$; then there exists a function $x \in D(L) \cap D(N)$ which is a solution of the equation $Lx = Nx$. Moreover $\| x - x_0 \| \leqslant d$ and $\mu(x-x_0) \leqslant \bar{R}$. The proof is a straight forward application of Brouwer's fixed point theorem.

4. An Illustrative Example

We make use of the theory developed in earlier sections and prove the existence of a solution to the following second order functional differential equation with periodic boundary conditions:

$$y'' + y = k[y^2(t - \pi/8) + y'(t - \pi/8)] + 0.03 \sin 2t, \ |k| \leqslant 0.5 \qquad \ldots(4.1)$$

$$y_0 = y_\pi, \ y'_0 = y'_\pi \qquad \ldots(4.2)$$

over the interval $J = [0, \pi]$. Here $\alpha = \pi/8$.

Putting the above equation in the system form, we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Big((k[x_1^2(t-(\pi/8)) + x_2^2(t-(\pi/8))] + 0.03 \sin 2t) \Big) \qquad \ldots(4.3)$$

$$(x_1)_0 = (x_1)_\pi, \ (x_2)_0 = (x_2)_\pi \qquad \ldots(4.4)$$

where

$$x_1 = y, \ x_2 = y'.$$

Here, we have

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ X(t, xt)$$

$$0$$

$$= \Big((k[x_1^2(t-(\pi/8)) + x_2^2(t-(\pi/8))] + 0.03 \sin 2t) \Big)$$

$$B(x) = x(0) - x(\pi).$$

Observe that

$$\Gamma x = x^{(1)} - A(t)x = x' - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x.$$

The operator L is defined by

$$D(L) = \{x \in H(J) : B(x) = x(0) - x(\pi) = 0\},$$

$$Lx = x' - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x. \qquad \ldots(4.5)$$

We check that the adjoint L^* of L is given by

$$D(L) = \{z \in H(J) : z(0) - z(\pi) = 0\},$$

$$Lz = z' - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z.$$

Clearly, dim $N(L)$ = dim $N(L) = 0 = p = q$ and dim $N(\ ) = 2$. After lengthy calculations it is shown that the problem (4.1) – (4.2) has a solution y on $[0, \pi]$. Moreover $|y(t)| < 0.1$ and $|y'(t)| < 0.1$ for all $t \in [0, \pi]$.

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