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A CHARGED PERFECT FLUID DISTRIBUTION OF CLASS-ONE

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In this paper a class-one model of a perfect fluid distribution under an electric field has been derived. The model of cylindrical symmetry which is obtained is of class-one but not conformal to flat space-time. It is rotation free but expansion and shear do not vanish. The lines of flow are geodesics and a particle which is initially at rest does not remain at rest. The coordinates chosen are comoving. The expression for the redshift is also obtained.

1. INTRODUCTION

The cylindrically symmetric line-element of Marder has been considered in various contexts by different authors and numerous solutions are already available. Recently some cosmological models of cylindrical symmetry have been obtained by Singh and Singh (1968), Singh and Abdussattar (1973) and Roy and Singh (1976) representing perfect fluid distribution. In the present paper we consider a class-one model of a perfect fluid distribution under an electric field. The coordinate system adopted is comoving. The model is rotation free but expansion and shear do not vanish. The motion of a charged particle is discussed and it is found that a particle which is initially at rest does not remain at rest. The red-shift has also been obtained.

2. THE FIELD EQUATIONS

We consider the cylindrically-symmetric metric in the form,

$$ds^2 = A^2 (dt^2 - dx^2) - B^2 dy^2 - C^2 dZ^2 \quad \ldots(2.1)$$

where A, B, C are functions of t alone.

The energy-momentum tensor for a charged perfect fluid distribution is given by

$$T_i^k = (\rho + p) V_i V^k - p\delta_i^k + E_i^k \quad \ldots(2.2)$$

where E_i^k is given by

$$E_i^k = - F^{ka} F_{ia} + \tfrac{1}{4} \delta_i^k F^{ab} F_{ab} \quad \ldots(2.3)$$

together with

$$F_{[ij,k]} = 0 \quad \ldots(2.4)$$

and

$$(\sqrt{-g}\, F^{ik})_{,k} = \sqrt{-g}\, J^{i}. \qquad \ldots(2.5)$$

V^i is the flow vector satisfying

$$V_i V^i = 1 \qquad \ldots(2.6)$$

p being the isotropic pressure and ρ the density. A comma (,) indicates partial differentiation.

We consider only purely electric case i.e. only F_{14} is the surviving components of F_{ik}.

The coordinates are assumed to be comoving so that

$V^1 = V^2 = V^3 = 0$ and $V^4 = 1/A$.

The field equations

$$R_i^k - \tfrac{1}{2} R\delta_i^k + \Lambda\, \delta_i^k = -8\pi T_i^k \qquad \ldots(2.7)$$

for the line-element (2.1) are as follows,

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} - \frac{B_4C_4}{BC} - \frac{B_{44}}{B} - \frac{C_{44}}{C}\right] + \Lambda = 8\pi p + 4\pi\, F^{14}\, F_{14} \qquad \ldots(2.8)$$

$$\frac{1}{A^2}\left[\frac{A_4^2}{A^2} - \frac{A_{44}}{A} - \frac{C_{44}}{C}\right] + \Lambda = 8\pi p - 4\pi\, F^{14}\, F_{14} \qquad \ldots(2.9)$$

$$\frac{1}{A^2}\left[\frac{A_4^2}{A^2} - \frac{A_{44}}{A} - \frac{B_{44}}{B}\right] + \Lambda = 8\pi p - 4\pi\, F^{14}\, F_{14} \qquad \ldots(2.10)$$

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC}\right] - \Lambda = 8\pi\rho - 4\pi\, F^{14}\, F_{14}. \qquad \ldots(2.11)$$

The suffix 4 after the symbols A, B, C denotes the ordinary differentiation with respect to t.

As the number of equations is less than the number of unknowns, for complete solution of eqns. (2.8)-(2.11) we need an extra condition. We choose the condition so that the line-element (2.1) may be of class-one. The condition is given by

$$R_{hijk} = e\,(b_{ik}\, b_{hj} - b_{ij}\, b_{hk}) \qquad \ldots(2.12)$$

and

$$b_{ij;k} - b_{ik;j} = 0 \qquad \ldots(2.13)$$

where b_{ij} are the coefficients of the second fundamental form.

From eqns. (2.8) and (2.9) we get,

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} - \frac{B_4C_4}{BC} - \frac{B_{44}}{B} - \left(\frac{A_4}{A}\right)^2 + \frac{A_{44}}{A}\right] = 8\pi\, F^{14}\, F_{14}\,. \qquad \ldots(2.14)$$

Also from eqns. (2.9) and (2.10) we get,

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = 0. \qquad \ldots(2.15)$$

The conditions (2.12) and (2.13) when applied to (2.1) lead to

$$\left.\begin{aligned} C_4B_{44} - C_{44}B_4 &= 0 \\ B_4A_{44} - B_{44}A_4 &= 0 \end{aligned}\right\}. \qquad \ldots(2.16)$$

Equations (2.16) ultimately yield the solution

$$A = k_1C + k_2 \qquad \ldots(2.17)$$

$$B = k_3C + k_4.$$

From eqns. (2.15) and (2.17) we have

$$C = k_5t + k_6 \qquad \ldots(2.18)$$

where k_1, k_2, k_3, k_4, k_5 and k_6 are arbitrary constants.

Equations (2.17) and (2.18) lead to

$$A = k_1k_5t + k_1k_6 + k_2 \qquad \ldots(2.19)$$

$$B = k_3k_5t + k_3k_6 + k_4\,. \qquad \ldots(2.20)$$

By a suitable transformation of coordinates the metric (2.1) can be put into the form,

$$ds^2 = (1 + \alpha T)^2 (dT^2 - dX^2) - (1 + \beta T)^2 dY^2 - (1 + \gamma T)^2 dZ^2 \qquad \ldots(2.21)$$

where α, β and γ arbitrary constants.

The component F^{14} of the electromagnetic field tensor for the metric (2.21) is given by

$$F^{14} = \frac{1}{2\sqrt{2\pi}\,(1+\alpha T)^3}\left[\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)} - \frac{\alpha\beta}{(1+\alpha T)(1+\beta T)} - \frac{\alpha\gamma}{(1+\alpha T)(1+\gamma T)}\right]^{1/2}. \qquad \ldots(2.22)$$

The pressure p and density ρ are as follows :

$$8\pi p = \frac{1}{2(1+\alpha T)^2}\left[\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\alpha\beta}{(1+\alpha T)(1+\beta T)} + \frac{\alpha\gamma}{(1+\alpha T)(1+\gamma T)} - \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)}\right] + \Lambda \qquad \ldots(2.23)$$

and

$$8\pi\rho = \frac{1}{2(1+\alpha T)^2}\left[\frac{3\alpha\beta}{(1+\alpha T)(1+\beta T)} + \frac{3\alpha\gamma}{(1+\alpha T)(1+\gamma T)} + \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)} - \frac{\alpha^2}{(1+\alpha T)^2}\right] - \Lambda. \qquad \ldots(2.24)$$

Reality conditions $p > 0$, $\rho > 0$, $\rho \geqslant 3p$ lead to

$$\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\alpha\beta}{(1+\alpha T)(1+\beta T)} + \frac{\alpha\gamma}{(1+\alpha T)(1+\gamma T)} + 2\Lambda(1+\alpha T)^2 > \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)}$$

$$\frac{3\,\alpha\beta}{(1+\alpha T)(1+\beta T)} + \frac{3\,\alpha\gamma}{(1+\alpha T)(1+\gamma T)} + \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)} > \frac{\alpha^2}{(1+\alpha T)^2} + 2\Lambda(1+\alpha T)^2$$

(*equation continued on p.* 1310)

$$\frac{\beta\gamma}{(1+\beta T)(1+\gamma T)} \geqslant \frac{\alpha^2}{(1+\alpha T)^2} + 2\Lambda (1+\alpha T)^2 .$$

The non-vanishing component of the current vector is given by,

$$J^1 = \frac{1}{2\sqrt{2\pi}}\left[\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)} - \frac{\alpha\beta}{(1+\alpha T)(1+\beta T)} - \frac{\alpha\gamma}{(1+\alpha T)(1+\gamma T)}\right]^{\frac{1}{2}}$$
$$\times\left[\frac{\gamma}{(1+\alpha T)^3(1+\gamma T)} + \frac{\beta}{(1+\alpha T)^3(1+\beta T)} - \frac{\alpha}{(1+\alpha T)^4}\right]$$
$$+\left\{\frac{\alpha^2}{(1+\alpha T)^2}\left(\frac{\beta}{1+\beta T} + \frac{\gamma}{1+\gamma T}\right) + \frac{\beta^2}{(1+\beta T)^2}\left(\frac{\alpha}{1+\alpha T} - \frac{\gamma}{1+\gamma T}\right)\right.$$
$$\left. + \frac{\gamma^2}{(1+\gamma T)^2}\left(\frac{\alpha}{1+\alpha T} - \frac{\beta}{1+\beta T}\right) - \frac{2\alpha^3}{(1+\alpha T)^3}\right\}\Big/ 4\sqrt{2\pi}\,(1+\alpha T)^3\left[\frac{\alpha^2}{(1+\alpha T)^2}\right.$$
$$\left. + \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)} - \frac{\alpha\beta}{(1+\alpha T)(1+\beta T)} - \frac{\alpha\gamma}{(1+\alpha T)(1+\gamma T)}\right]^{1/2} . \quad \ldots(2.25)$$

The rotation ω vanishes identically. The expressions for expansion θ and shear tensor σ_{ij} calculated for the flow vector V_i for the metric (2.21) are given by,

$$\theta = \frac{\alpha}{(1+\alpha T)^2} + \frac{\beta}{(1+\alpha T)(1+\beta T)} + \frac{\gamma}{(1+\alpha T)(1+\gamma T)} . \quad \ldots(2.26)$$

$$\sigma_{11} = \frac{1+\alpha T}{3}\left(-\frac{2\alpha}{1+\alpha T} + \frac{\beta}{1+\beta T} + \frac{\gamma}{1+\gamma T}\right) \quad \ldots(2.27)$$

$$\sigma_{22} = -\frac{2\beta(1+\beta T)}{3(1+\alpha T)} + \frac{\alpha(1+\beta T)^2}{3(1+\alpha T)^2} + \frac{\gamma(1+\beta T)^2}{3(1+\alpha T)(1+\gamma T)} \quad \ldots(2.28)$$

$$\sigma_{33} = -\frac{2\gamma(1+\gamma T)}{3(1+\alpha T)} + \frac{\alpha(1+\gamma T^2)}{3(1+\alpha T)^2} + \frac{\beta(1+\gamma T)^2}{3(1+\alpha T)(1+\beta T)} \quad \ldots(2.29)$$

the other components of the shear tensor σ_{ij} are zero.

The non-vanishing components of Weyl conformal curvature tensor C_{hijk} are as follows :

$$C^{14}_{14} = C^{23}_{23} = \frac{1}{3(1+\alpha T)^2}\left[\frac{\alpha^2}{(1+\alpha T)^2} - \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)}\right] \quad \ldots(2.30)$$

$$C^{12}_{12} = C^{34}_{34} = \frac{1}{6(1+\alpha T)^2}\left[-\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)}\right.$$
$$\left. + \frac{3\alpha(\gamma-\beta)}{(1+\alpha T)(1+\beta T)(1+\gamma T)}\right] \quad \ldots(2.31)$$

$$C^{13}_{13} = C^{24}_{24} = \frac{1}{6(1+\alpha T)^2}\left[-\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\beta\gamma}{(1+\beta T)(1+\gamma T)}\right.$$
$$\left. + \frac{3\alpha(\beta-\gamma)}{(1+\alpha T)(1+\beta T)\,1+\gamma T)}\right] . \quad \ldots(2.32)$$

The flow vector V^i is given by

$$V^1 = V^2 = V^3 = V_1 = V_2 = V_3 = 0$$

$$V^4 = \frac{1}{1+\alpha T},$$

and

$$V_4 = 1+\alpha T\,.$$

Clearly $V^i\ V^j_{;i} = 0$ so that the lines of flow are geodesics.

3. Motion of a Charged Particle

The motion of a charged particle is given by

$$\frac{d^2x^\mu}{dS^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dS}\frac{dx^\beta}{dS} + \frac{e}{m_0}F^\mu_\alpha\frac{dx^\alpha}{dS} = 0 \qquad ...(3.1)$$

where e/m_0 is the ratio of the charge particle to its rest mass and $F^\mu_\alpha = g_{\alpha\iota}F^{\iota\mu}$ is the electromagnetic field tensor.

For (2.21) equations (3.1) take the form

$$\frac{d^2X}{dS^2} + \frac{2\alpha}{1+\alpha T}\frac{dX}{dS}\frac{dT}{dS} - \frac{e}{m_0}\frac{1}{2\sqrt{2\pi}\,(1+\alpha T)}\left[\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\beta\gamma}{(1+\beta T)\,(1+\gamma T)} - \frac{\alpha\beta}{(1+\alpha T)\,(1+\beta T)} - \frac{\alpha\gamma}{(1+\alpha T)\,(1+\gamma T)}\right]^{1/2}\cdot\frac{dt}{ds} = 0 \qquad ...(3.2)$$

$$\frac{d^2Y}{dS^2} + \frac{2\beta}{1+\beta T}\frac{dY}{dS}\cdot\frac{dT}{dS} = 0 \qquad ...(3.3)$$

$$\frac{d^2Z}{dS^2} + \frac{2\gamma}{1+\gamma T}\frac{dZ}{dS}\frac{dT}{dS} = 0 \qquad ...(3.4)$$

$$\frac{d^2T}{dS^2} + \frac{\alpha}{1+\alpha T}\left(\frac{dX}{dS}\right)^2 + \frac{\beta\,(1+\beta T)}{(1+\alpha T)^2}\left(\frac{dY}{dS}\right)^2 + \frac{\gamma\,(1+\gamma T)}{(1+\alpha T)^2}\left(\frac{dZ}{dS}\right)^2 + \frac{\alpha}{1+\alpha T}\left(\frac{dT}{dS}\right)^2 - \frac{e}{m_0}\frac{1}{2\sqrt{2\pi}\,(1+\alpha T)}\left[\frac{\alpha^2}{(1+\alpha T)^2} + \frac{\beta\gamma}{(1+\beta T)\,(1+\gamma T)} - \frac{\alpha\beta}{(1+\alpha T)\,(1+\beta T)} - \frac{\alpha\gamma}{(1+\alpha T)\,(1+\gamma T)}\right]^{1/2}\cdot\frac{dX}{dS} = 0. \qquad ...(3.5)$$

If the particle is initially at rest, i.e., if

$$\frac{dX}{dS} = \frac{dY}{dS} = \frac{dZ}{dS} = 0$$

we get

$$\frac{dT}{dS} = \frac{1}{1+\alpha T}\,.$$

From eqns. (3.2) to (3.4) we find that for all such particles the components of spatial acceleration are given by

$$\left.\begin{aligned}\frac{d^2X}{dS^2} &= \frac{e}{m_0}\frac{1}{2\sqrt{2\pi}\,(1+\alpha T)^2}\left[\frac{\alpha^2}{(1+\alpha T)^2}+\frac{\beta\gamma}{(1+\beta T)(1+\gamma T)}\right.\\ &\qquad\left.-\frac{\alpha\beta}{(1+\alpha T)(1+\beta T)}-\frac{\alpha\gamma}{(1+\alpha T)(1+\gamma T)}\right]^{1/2}\\ \frac{d^2Y}{dS^2} &= 0,\ \frac{d^2Z}{dS^2}=0\,.\end{aligned}\right\} \quad \ldots(3.6)$$

Hence the particle would not remain permanently at rest.

4. Red Shift in the Field

Following the method outlined by Tolman (1962), the red shift in this case is given by

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{(1+\alpha T)}{(1+\alpha T_2)\,(1+\gamma T_2)^{-1}} \times \frac{\{(1+\alpha T_1)\,(1+\gamma T_1)^{-1}+u_z\}}{[(1+\alpha T)^2-u^2]^{1/2}} \quad \ldots(4.1)$$

where u_z is the component of the velocity of the particle along the z-axis at the time of emission and u is the velocity of the source at the time of emission.

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