Indian J. pure appl. Math., **13**(11) : 1313–1321, November 1982

BENDING OF A SANDWICH ANNULAR PLATE OF VARIABLE THICKNESS

ANAND P. GUPTA* AND KAMESHWAR P. SHARMA

Department of Mathematics, University of Roorkee, Roorkee 247672

(*Received* 8 *May* 1981)

The problem of axisymmetric bending of sandwich annular plates having a core of linearly varying thickness and facings of constant thickness is solved by a splines-technique. The numerical results are obtained by varying the various parameters in such a way that total volume of the plate and the total load on it remain constant.

1. INTRODUCTION

Nowadays, in order to achieve lightness with strength, increasing use of sandwich construction is being made in missiles and spacecraft structures. Often, in such applications the thickness of the plate is a function of radial distance from the centre. By tapering the plate thickness, the weight: stiffness ratio would improve relative to uniform thickness plate and possibly be sufficient to offset any extra fabrication costs. Fixed volume of the plate is important for design of structures, as well as for the comparison of different tapers.

An account of work done upto 1965, is given by Habip (1965) and Plantemma (1966). Kao (1965), has derived the system of equations for sandwich circular plate considering facings as membranes using Largrangian multipliers, but no numerical results are presented. Stickney and Abdulhadi (1968), have presented small deflection theory for the analysis of orthotropic circular sandwich plates by extremizing the complementary strain energy. Raggett *et al.* (1974), have used cubic splines for circular annulus of varying thickness.

In the present investigation, the axisymmetric bending of sandwich annular plate of variable thickness is considered. Transverse shear is taken into account for the core, but not for the facings, which are assumed to be membranes. The core thickness is assumed to vary linearly with respect to the radial co-ordinate, but the facings are taken of constant thickness. The equations of equilibrium are derived by using the principle of minimum potential energy and solved by a splines-technique. Numerical results for transverse deflection and shear forces are computed for uniform and linearly varying loads for various values of core and facings thickness, inner radius, and the taper and load parameters β and γ. In each case, the total volume and total load on the plate are kept unchanged.

*Present Address : Dr A. P. Gupta, Department of Mechanical Engg., Military Technical College, Rashid Camp, Post Box 478, Baghdad, Iraq.

2. Formulation of the Problem

We consider a sandwich annular plate of thickness $2h$, and outer and inner radii a and b respectively. The thickness of the core and each of the two facings are taken $2h_1$ and h_2, so that $h = h_1 + h_2$. The thickness of the core varies with respect to the radial co-ordinate, whereas that of the facings remians constant. The plate is referred to cylindrical co-ordinates r, θ and z by taking the axis of the plate as line $r = 0$; middle plane, lower and upper interfacings and bottom and top as the surfaces $z = 0, \pm h_1, \pm h$. The two facings are made of same material, which is different from that of the core. Therefore, the various quantities will be distinguished by subscripts 1, 2 and 3 respectively.

Since we are considering the axisymmetric bending of the plate and the facings are assumed to be membranes, the displacements in the core and facings are approximated as

$$\left.\begin{aligned} &u_1(r, z) = z\psi_1(r);\ u_2(r, z) = -u_3(r, z) = h_1(r)\psi_1(r) \\ &v_i(r, z) = 0;\ w_i(r, z) = w(r); \end{aligned}\right\} \quad \ldots(1)$$

where $i = 1, 2, 3$.

In the above expression, u_i, v_i and w_i are the displacements in the directions of r, θ and z respectively, ψ_1 is the rotation in rz-plane of the normal to the middle plane of the plate for the core.

The non-zero strain components are found to be

$$\left.\begin{aligned} &\epsilon_{r1} = z\psi_1(r);\ \epsilon_{r2} = -\epsilon_{r3} = h_{1,r}\psi_1 + h_1\psi_{1,r}; \\ &\epsilon_{\theta 1} = z\psi_1/r;\ \epsilon_{\theta 2} = -\epsilon_{\theta 3} = h_1\psi_1/r; \\ &\epsilon_{rz_1} = \psi_1 + w_{,r}; \end{aligned}\right\} \quad \ldots(2)$$

where a comma followed by a suffix denotes the differentation with respect to that variable.

The stress-strain relations are

$$\left.\begin{aligned} &\sigma_{ri} = \lambda_i(\epsilon_{ri} + \nu_i\epsilon_{\theta i});\ \sigma_{\theta_i} = \lambda_i(\epsilon_{\theta i} + \nu_i\,\epsilon_{r_i}); \\ &\sigma_{rzi} = \mu_i\epsilon_{rz_i};\ \mu_i = E_i/\{2(1 + \nu_i)\};\ \lambda_i = E_i/(1 - \nu^2_i): \end{aligned}\right\} \quad \ldots(3)$$

where E_i and ν_i are the Young's Modulii and Poisson's ratios respectively.

3 Equations of Equilibrium

The strain energy of an isotropic sandwich annular plate, neglecting the direct transverse strains ϵ_{zi}, may be written as

$$V = \sum_{i=1}^{3}\int_b^a\int_0^{2\pi}\int_{x_i}^{y_i}(\sigma_{r_i}\epsilon_{ri} + \sigma_{\theta_i}\epsilon_{\theta i} + \sigma_{rz_i}\epsilon_{rzi})\, r\, dz d\theta dr; \quad \ldots(4)$$

where the limits of integration, x_i to y_i $(i = 1, 2, 3)$, stand for $-h_1$ to h_1, h_1 to h, $-h$ to $-h_1$ respectively. Substituting the expressions for the strain components

(2), into (4) and then performing first two integrations, the strain energy of the plate may be expressed in the form

$$V = \pi \int_b^a [r\, P_1\psi_{1,r} + P_2\psi_1 + rQ_{r1}w_{,r}]\, dr \; ; \qquad ...(5)$$

where

$$P_1 = M_{r1} - h_1 (N_{r2} - N_{r3}) \; ;$$
$$P_2 = M_{\theta 1} + rQ_{r1} - rh_{1,r} (N_{r2} - N_{r3}) - h_1 (N_{\theta 2} - N_{\theta 3}).$$

The stress-resultants are given by

$$(N_{ri}, N_{\theta i}, M_{ri}, M_{\theta i}, Q_{ri}) = \int_{x_i}^{y_i} (\sigma_{ri}, \sigma_{\theta i}, z\sigma_{ri}, z\sigma_{\theta i}, k_s\, \sigma_{rzi})\, dz \; ; \qquad ...(6)$$

where limits of integration x_i, y_i are same as for eqn. (4). Since bending of facings and core is considered separately, we have taken the averaging shear constant $k_s = 1$ in our analysis.

The change in potential energy of the system due to applied surface load $p(r)$, is

$$U = - \int_b^a \int_0^{2\pi} p(r)\, wr\, d\theta\, dr. \qquad ...(7)$$

For a system in equilibrium, the first variation of the total potential energy vanishes for any arbitrary set of variations of the dependent variables, w and ψ_1, compatible with the prescribed boundary conditions, i.e.,

$$\delta T \equiv \delta U + \delta V = 0. \qquad ...(8)$$

Carrying out the first variation and integrating by parts those integrals containing derivatives of dependent variables w and ψ_1, we get

$$\int_b^a [(rP_{1,r} + P_1 - P_2)\, \delta\psi_1 + (rQ_{r1,r} + Q_{r1} + rp)\, \delta w]\, dr$$
$$- \pi\, [rP_1 \delta\psi_1 + rQ_{r1}\delta w]_a^b = 0. \qquad ...(9)$$

The coefficients corresponding to variations δw and $\delta\psi_1$ vanish, giving two equilibrium equations. The vanishing of expressions already integrated with respect to r, give the edge conditions.

With the help of (3) and (6), the equations of equilibrium are given in terms of displacements by

$$(rh_1h_{1,rr}\lambda_1 + rh_2h_{1,r}\lambda_2 - h_1h_2\lambda_2 - h_1^2\lambda_1/3)\, \psi_1 + (rh_1^2\lambda_1/3 + rh_1h_2\lambda_2$$
$$+ 2r^2h_2h_{1,r}\lambda_2 + r^2h_1h_{1,r}\lambda_1)\; \psi_{1,r} + (r^2h_1^2\lambda_1/3 + r^2h_1h_2\lambda_2)\, \psi_{1,rr}$$
$$- \mu_1\, (\psi_1 + w_{,r}) = 0 \; ; \qquad ...(10)$$

$$\mu_1\, (rh_{1,r} + h_1)\, (\psi_1 + w_{,r}) + 2\mu_1 h_1\, (\psi_{1,r} + w_{,rr}) + rp(r) = 0. \qquad ...(11)$$

The above equations are rendered dimensionless by using the following relationships:

$$\left.\begin{aligned} R &= r/a;\ R_0 = b/a;\ H_1 = h_1/a;\ H_2 = h_2/a; \\ G_1 &= \lambda_1/\mu_1;\ G_2 = \lambda_2/\mu_2;\ G_3 = \mu_2/\mu_1; \\ S_1 &= \mu_1\psi_1/P;\ W = \mu_1 w/aP;\ \bar{p}(R) = p(r)/P; \end{aligned}\right\} \quad \ldots(12)$$

where P is the average load per unit area.

Let the thickness of the core vary linearly as :

$$H_1 = H_0 - \beta(R - R_0)\,; \quad \ldots(13)$$

where $2H_0$ is the non-dimensional thickness of the core at $R = R_0$ and β is the taper parameter.

With the help of relations (12) and (13), eqns. (10) and (11) reduce to

$$B_1S_1 + B_2S_{1,R} + B_3S_{1,RR} + B_4W_{,R} = 0 \quad \ldots(14)$$

$$B_5(S_1 + W_{,R}) + B_6(S_{1,R} + W_{,RR}) = -R\,\bar{p}(R)/2 \quad \ldots(15)$$

where

$$A_1 = H_0(H_2G_2G_3 + H_0G_1/3)\,;\ A_2 = A_3 + 2A_4/3\,;$$

$$A_3 = H_2\beta G_2G_3\,;\ A_4 = H_0\beta G_1\,;\ A_5 = \beta^2G_1/3\,;$$

$$B_1 = -A_1 + A_2B_0 - (A_4\nu_1 + A_3)R - (B_0 - 3\nu_1R)A_5B_0 + B_4\,;$$

$$B_2 = R\{A_1 - (A_2 + 2A_3)B_0 - A_4R + (B_0 + 3R)A_5B_0\}\,;$$

$$B_3 = R^2\{A_1 - (A_2 - A_5B_0)B_0\}\,;\ B_4 = -R^2\,;\ B_0 = R - R_0.$$

The non-dimensional bending moments and shear forces are given by

$$\left.\begin{aligned} m_R &= M_{r1}/a^2P = 2H_1^3G_1(S_{1,R} + \nu_1S_1/R)/3\,; \\ m_\theta &= M_{\theta 1}/a^2P = 2H_1^3G_1(S_1/R + \nu_1S_{1,R})/3\,; \\ q_R &= Q_{r1}/aP = 2H_1(S_1 + W_{,R}). \end{aligned}\right\} \quad \ldots(16)$$

4. Solution Using Splines

For solving the equations of equilibrium (14) and (15), the splines technique is adopted. Raggett *et al.* (1974), have approximated a single function by splines. In a similar fashion, the functions S_1 and W can be represented by the cubic splines $\bar{S}_1(R)$ and $\bar{W}(R)$, over the interval $(R_0, 1)$ by inserting the knots at points $R_0, R_1, R_2, \ldots R_n$, where $b/a = R_0 < R_1 < R_2 < \ldots < R_n = 1$.

The functions $\bar{S}_1(R)$ and $\bar{W}(R)$ can be taken as :

$$\left.\begin{aligned} \bar{S}_1(R) &= a_0 + a_1(R - R_0) + a_2(R - R_0)^2 + \sum_{j=0}^{n-1} b_j(R - R_j)_*^3\,; \\ \bar{W}(R) &= c_0 + c_1(R - R_0) + c_2(R - R_0)^2 + \sum_{j=0}^{n-1} b_j(R - R_j)_*^3\,; \end{aligned}\right\} \quad \ldots(17)$$

where
$$(R - R_j)_* = \begin{cases} (R - R_j)\,; & \text{for } R \geqslant R_j \\ 0\,; & \text{for } R < R_j \end{cases}.$$

For inner and outer edge clamped, the boundary conditions are

$$\psi_1(a) = \psi_1(b) = w(a) = w(b) = 0. \qquad \ldots(18)$$

The substitution of eqns. (17) in eqns. (14) and (15), gives

$$\begin{aligned} a_0B_1 &+ a_1\{B_1(R-R_0)+B_2\} + a_2\{B_1(R-R_0)^2 + 2B_2(R-R_0) + 2B_3\} \\ &+ \sum_{j=0}^{n-1} b_j(R-R_j)_*\{B_1(R-R_j)_*^2 + 3B_2(R-R_j)_*\} + c_1B_4 + c_2\,2B_4(R-R_0) \\ &+ 3B_4\sum_{j=0}^{n-1} d_j(R-R_j)_*^2 = 0\,; \qquad \ldots(19) \end{aligned}$$

$$\begin{aligned} a_0B_5 &+ a_1\{B_5(R-R_0)+B_6\} + a_2\{B_5(R-R_0)^2 + 2B_6(R-R_0)\} \\ &+ \sum_{j=0}^{n-1} b_j(R-R_j)_*^2\{B_5(R-R_j)_* + 3B_6\} + c_1B_5 + c_2\{2B_5(R-R_0) + 2B_6\} \\ &+ \sum_{j=0}^{n-1} d_j(R-R_j)_*\{3B_5(R-R_j)_* + 6B_6\} = -R\bar{p}(R)/2. \qquad \ldots(20) \end{aligned}$$

The satisfaction of the above equations at the knots $R_0, R_1, R_2, \ldots R_n$ together with edge conditions (18), give a set of $2(n+3)$ linear simultaneous equations to be solved for the unknowns a_i, b_i $(i = 0, 1, 2)$, and c_i, d_i $(i = 0, 1, 2, \ldots, n-1)$. It is convenient to take the knots equally spaced in the interval $(R_0, 1)$. Therefore, we take

$$R_j = R_0 + j\,\Delta R\,;$$

where $j = 0, 1, \ldots, n-1$ and ΔR is the distance between successive knots.

Volume of the Plate

Let H_f be the thickness of the facings of the plate perpendicular to the inter-facings, then

$$H_f = H_2/\sqrt{1-\beta^2}, \qquad \ldots(21)$$

If the variations in β, H_0, R_0 and H_f are taken in such a way that the total volume V_p of the plate remains unchanged, then

$$\tfrac{1}{2}V_p = \pi H_m(1-R_0^2) = \pi\int_{R_0}^{1}\{H_0 + H_2 - \beta(R-R_0)\}\,R\,dR \qquad \ldots(22)$$

where $2H_m$ is the average thickness of the plate.

Uniform Loading

The total load on the plate is $\pi P(1-R_0^2)$. Therefore, for a uniformly distributed load, we take

$$\bar{p}(R) = 1. \qquad \ldots(23)$$

Variable Loading

For a load varying linearly along the radius of the plate, we take

$$\bar{p}(R) = p_0\{1-\gamma(R-R_0)\} \qquad \ldots(24)$$

where p_0 is the load intensity at $R=R_0$ and γ is the load parameter. If we vary the load in such a manner that the total load on the plate remains unchanged, then

$$\int_{R_0}^{1} \int_{0}^{2\pi} p_0 \{1-\gamma (R-R_0)\} R d\theta dR = P\pi (1-R_0^2). \qquad \ldots(25)$$

Therefore,

$$p_0/P = 3 (1+R_0)/\{(3+\gamma R_0)(1+R_0) - 2\gamma\}. \qquad \ldots(26)$$

5. Numerical Results and Discussion

The material for the core and of the facings are taken to be cellular cellulose acetate and aluminium, for which the values of various constants are

$G_1 = 2.20$; $G_2 = 2.8461$; $G_3 = 1683.0$; $\nu_1 = 0.0909$.

The transverse deflection W and shear forces q_R are computed and plotted for various values of H_f, β, R_0 and γ by keeping the volume of the plate and total load on the plate constant. The volume of the plate is kept constant by taking $H_m = 0.1$. To choose the appropriate interpolation interval ΔR, the computer programme developed for the evaluation of transverse deflection W was run for $n = 10\ (5)\ 35$. The numerical values in Table I, show a consistent improvement with the increase in number of knots. In our remaining computations, we have taken $n = 25$, since further increase in n does not improve the results except in the fourth or fifth decimal place.

Table I

Transverse deflection W for various values of n.

$H_f = 0.002$, $\beta = 0.04$, $R_0 = 0.2$

n	Transverse Deflection W		
	R		
	0.4	0.6	0.8
10	0.3660	0.4364	0.3081
15	0.3632	0.4349	0.3071
20	0.3623	0.4338	0.3068
25	0.3620	0.4335	0.3067
30	0.3618	0.4334	0.3066
35	0.3617	0.4333	0.3066

The behaviour of W with H_f for uniform load is given in Table II for various values of R and $\beta = 0.04$. The table shows that there is an abrupt decrease in W, when we go from homogeneous plate to sandwich plate, even when the facings are very small. W decreases slowly when the facings thickness increases. After a certain value of H_f, W increases slightly with increase in H_f. The first two results follow

TABLE II

Transverse deflection W for uniform loading for various values of H_f, $\beta = 0.04$, $R_0 = 0.2$.

H_f $R-$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	0.3971	0.7857	1.0477	1.1210	0.9892	0.6846	0.2988
0.0002	0.2456	0.3932	0.4666	0.4754	0.4258	0.3241	0.1784
0.0004	0.2368	0.3746	0.4418	0.4500	0.4053	0.3122	0.1753
0.0006	0.2377	0.3682	0.4334	0.4415	0.3986	0.3085	0.1754
0.0008	0 2322	0.3652	0.4295	0.4375	0.3955	0.3070	0.1743
0.0010	0 2315	0.3636	0.4274	0.4355	0.3940	0.3063	0.1744
0 0020	0.2313	0.3624	0.4255	0.4339	0.3933	0.3068	0.1756
0.0030	0.2326	0.3643	0.4277	0 4362	0.3959	0.3093	0.1774
0.0040	0.2345	0.3671	0.4310	0.4397	0.3993	0.3122	0.1792
0.0050	0.2365	0.3703	0.4348	0.4437	0.4031	0.3154	0.1812

from physical consideration also, while the third appears to be due to facings being considered as membranes. It will be more natural that the deflection decreases with the increase in the stronger material.

Deflection profiles for uniform loading for various values of β are given in Fig. 1. It shows that W increases in the inner half of the plate and decreases in the outer half of the plate with decrease in β. The decrease in β also shows an increase in the maximum deflection and a shift in the point of maximum deflection towards the inner side of the plate. Deflection profiles for various values of R_0 are given in Fig. 2. The figure shows that W decreases with the increase in R_0. Deflection profiles for linearly varying load are given in Fig. 3, for various values of γ, which shows

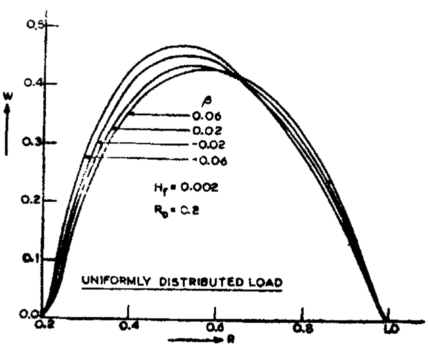

FIG. 1. Deflection profiles for various values of β.

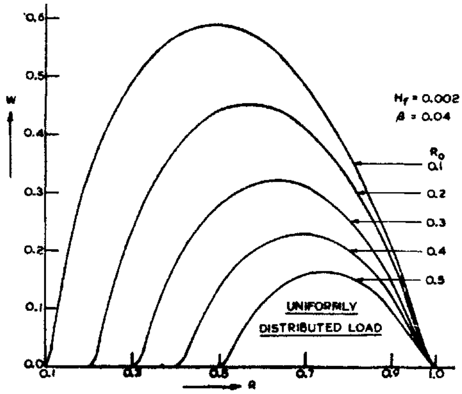

FIG. 2. Deflection profiles for various values of R_0

that W decreases with the decrease in γ and the point of maximum deflection shifts slightly towards the outer edge of the plate. All these results are supported by physical considerations.

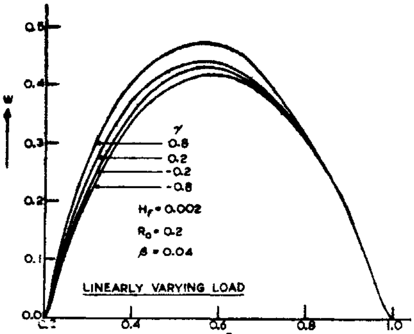

FIG. 3. Deflection profiles for various values of γ.

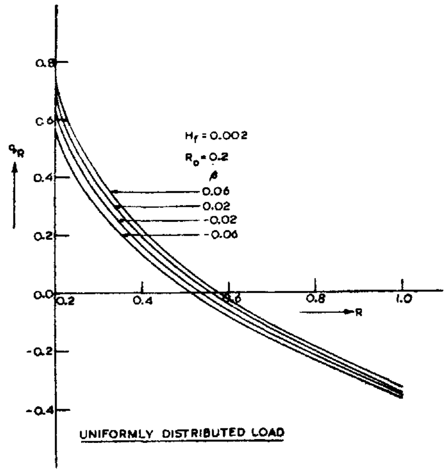

FIG. 4. q_R vs. R for various values of β.

q_R versus R for various values of β and R_0 for uniform loading and for various values of γ are plotted in Figs. 4, 5 and 6. Figure 4 shows that q_R decreases constantly with the decrease in β. Figure 5 shows that q_R at R_0 increases as R_0 decreases. Figure 6 shows that q_R first decreases, then increases and finally decreases with the decrease in γ.

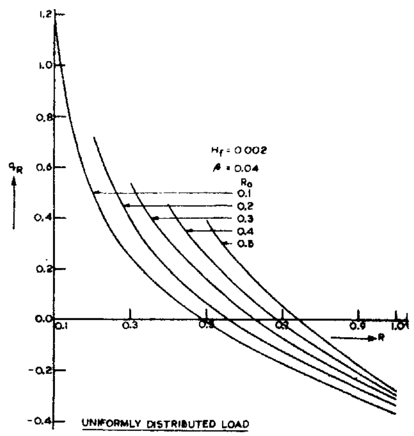

FIG. 5. q_R vs R for various values of R_0.

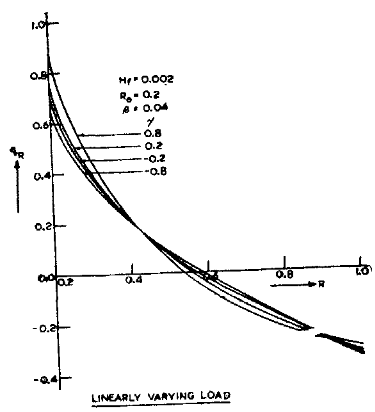

FIG. 6. q_R vs R for various values of γ.

ACKNOWLEDGEMENT

The authors are grateful to Dr C. Prasad for his encouragement and helpful comments. The authors they would like to thank Roorkee University, Regional Computer Centre for computational work. The financial support of University Grants Commission to the second author is duly acknowledged.

REFERENCES

Habip, L. M. (1965). A survey of modern developments in the analysis of sandwich structures. *AMR*, **18** (2), 93–95.

Kao, J. S. (1965). Bending of Circular sandwich plates. *J. Engg. Mech. Dn., ASCE, EM4*, 165–76.

Plantemma, F. J. (1966). Sandwich Construction. John Wiley and Sons, Inc., New York.

Raggett, G. E. *et al.* (1974). On the use of cubic splines to solve certain circular plate problem. *Computer Methods Appl. Mech. Engg.*, **4**, 39–45.

Stickney, G. H., and Abdulhadi, F. (1968). Flexure of multilayer orthotropic circular sandwich plates. *J. Composite Material*, **2**, 209–19.